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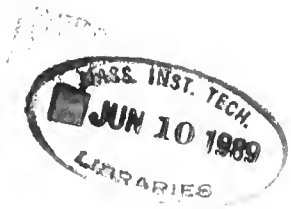

NETWORK FLOWS

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NETWORK FLOWS

OVERVIEW

Network Flows

Perhaps no subfield of mathematical programming is more alluring than network optimization. Highway, rail, electrical, communication and many other physical networks pervade our everyday lives. As a consequence, even non-specialists recognize the practical importance and the wide ranging applicability of networks. Moreover, because the physical operating characteristics of networks (e.g., flows on arcs and mass balance at nodes) have natural mathematical representations, practitioners and non-specialists can readily understand the mathematical descriptions of network optimization problems and the basic nature of techniques used to solve these problems. This combination of widespread applicability and ease of assimilation has undoubtedly been instrumental in the evolution of network planning models as one of the most widely used modeling techniques in all of operations research and applied mathematics.

Network optimization is also alluring to methodologists. Networks provide a concrete setting for testing and devising new theories. Indeed, network optimization has inspired many of the most fundamental results in all of optimization. For example, price directive decomposition algorithms for both linear programming and combinatorial optimization had their origins in network optimization. So did cutting plane methods and branch and bound procedures of integer programming, primal-dual methods of linear and nonlinear programming, and polyhedral methods of combinatorial optimization. In addition, networks have served as the major prototype for several theoretical domains (for example, the field of matroids) and as the core model for a wide variety of min/max duality results in discrete mathematics.

Moreover, network optimization has served as a fertile meeting ground for ideas from optimization and computer science. Many results in network optimization are routinely used to design and evaluate computer systems, and ideas from computer science concerning data structures and efficient data manipulation have had a major impact on the design and implementation of many network optimization algorithms.

The aim of this paper is to summarize many of the fundamental ideas of network optimization. In particular, we concentrate on network flow problems and highlight a number of recent theoretical and algorithmic advances. We have divided the discussion into the following broad major topics:

- Applications
- Basic Properties of Network Flows
- Shortest Path Problems
- Maximum Flow Problems
- Minimum Cost Flow Problems
- Assignment Problems

Much of our discussion focuses on the design of provably good (e.g., polynomial-time) algorithms. Among good algorithms, we have presented those that are simple and are likely to be efficient in practice. We have attempted to structure our discussion so that it not only provides a survey of the field for the specialists, but also serves as an introduction and summary to the non-specialists who have a basic working knowledge of the rudiments of optimization, particularly linear programming.

In this chapter, we limit our discussions to the problems listed above. Some important generalizations of these problems such as (i) the generalized network flows; (ii) the multicommodity flows; and (iv) the network design, will not be covered in our survey. We, however, briefly describe these problems in Section 6.6 and provide some important references.

As a prelude to the remainder of our discussion, in this section we present several important preliminaries . We discuss (i) different ways to measure the performance of algorithms; (ii) graph notation and various ways to represent networks quantitively; (iii) a few basic ideas from computer science that underlie the design of many algorithms; and (iv) two generic proof techniques that have proven to be useful in designing polynomial-time algorithms.

1.1 Applications

Networks arise in numerous application settings and in a variety of guises. In this section, we briefly describe a few prototypical applications. Our discussion is intended to illustrate a range of applications and to be suggestive of how network flow problems arise in practice; a more extensive survey would take us far beyond the scope of our discussion. To illustrate the breadth of network applications, we consider some models requiring solution techniques that we will not describe in this chapter.

For the purposes of this discussion, we will consider four different types of networks arising in practice:

- Physical networks (Streets, railbeds, pipelines, wires)
- Route networks
- Space-time networks (Scheduling networks)
- Derived networks (Through problem transformations)

These four categories are not exhaustive and overlap in coverage. Nevertheless, they provide a useful taxonomy for summarizing a variety of applications.

Network flow models are also used for several purposes:

- Descriptive modeling (answering "what is?" questions)
- Predictive modeling (answering "what will be?" questions)
- Normative modeling (answering "what should be?" questions, that is, performing optimization)

We will illustrate models in each of these categories. We first introduce the basic underlying network flow model and some useful notation.

The Network Flow Model

Let G = (N, A) be a directed network with a cost c_{ij}, a lower bound l_{ij}, and a capacity u_{ij} associated with every arc (i, j) $\in$ A. We associate with each node i $\in$ N an integer number b(i) representing its supply or demand. If b(i) > 0, then node i is a *supply* node; if b(i) < 0, then node i is a *demand* node; and if b(i) = 0, then node i is a *transhipment* node. Let n = | N | and m = | A |. The minimum cost network flow problem can be formulated as follows:

$$\text{Minimize} \sum_{(i,j) \in A} c_{ij} x_{ij} \tag{1.1a}$$

subject to

$$\sum_{\{j : (i,j) \in A\}} x_{ij} - \sum_{\{j : (j,i) \in A\}} x_{ji} = b(i), \quad \text{for all } i \in N, \tag{1.1b}$$

$$l_{ij} \le x_{ij} \le u_{ij}, \text{ for all } (i,j) \in A. \tag{1.1c}$$

We refer to the vector $x = (x_{ij})$ as the *flow* in the network. The constraint (1.1b) implies that the total flow out of a node minus the total flow into that node must equal

the net supply/demand of the node. We henceforth refer to this constraint as the *mass balance constraint*. The flow must also satisfy the lower bound and capacity constraints (1.1c) which we refer to as the *flow bound constraints*. The flow bounds might model physical capacities, contractual obligations or simply operating ranges of interest. Frequently, the given lower bounds l_{ij} are all zero; we show later that they can be made zero without any loss of generality.

In matrix notation, we represent the minimum cost flow problem

$$\text{minimize } \{ cx : Nx = b \text{ and } l \le x \le u \}, \tag{1.2}$$

in terms of a *node-arc incidence matrix* N. The matrix N has one row for each node of the network and one column for each arc. We let N_{ij} represent the column of N corresponding to arc (i, j), and let e_j denote the *j-th unit vector* which is a column vector of size n whose entries are all zeros except for the j-th entry which is a 1. Note that each flow variable x_{ij} appears in two mass balance equations, as an outflow from node i with a +1 coefficient and as an inflow to node j with a –1 coefficient. Therefore the column corresponding to arc (i, j) is $N_{ij} = e_i - e_j$.

The matrix N has very special structure: only 2m out of its nm total entries are nonzero, all of its nonzero entries are +1 or –1, and each column has exactly one +1 and one –1. Figure 1.1 gives an example of the node-arc incidence matrix. Later in Sections 2.2 and 2.3, we consider some of the consequences of this special structure. For now, we make two observations.

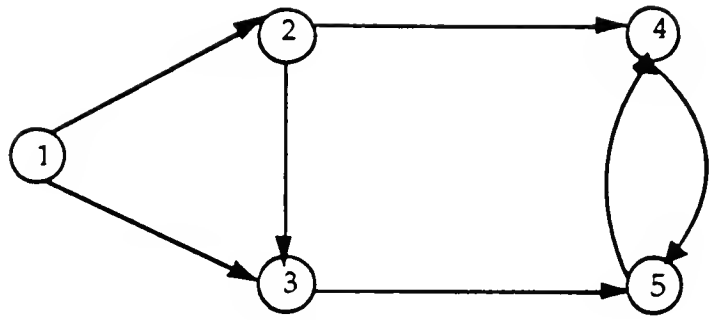
(i) Summing all the mass balance constraints eliminates all the flow variables and gives

$$\sum_{i \in N} b(i) = 0, \text{ or } \sum_{i \in \{N : b(i) > 0\}} b(i) = \sum_{i \in \{N : b(i) < 0\}} b(i) .$$

Consequently, total supply must equal total demand if the mass balance constraints are to have any feasible solution.

(ii) If the total supply does equal the total demand, then summing all the mass balance equations gives the zero equation 0x = 0, or equivalently, any equation is equal to minus the sum of all other equations, and hence is redundant.

The following special cases of the minimum cost flow problem play a central role in the theory and applications of network flows.



(a) An example network.

	(1, 2)	(1, 3)	(2, 3)	(2, 4)	(3, 5)	(4, 5)	(5, 4)
1	1	1	0	0	0	0	0
2	−1	0	1	1	0	0	0
3	0	−1	−1	0	1	0	0
4	0	0	0	−1	0	1	−1
5	0	0	0	0	−1	−1	1

(b) The node-arc incidence matrix of the example network

Figure 1.1. An example of the matrix N.

The shortest path problem. The shortest path problem is to determine directed paths of smallest cost from a given node 1 to all other nodes. If we choose the data in the minimum cost flow problem as b(1) = (n − 1), b(i) = −1 for all other nodes, and $l_{ij} = 0$ and $u_{ij} = n$ for all arcs, then the optimum solution sends unit flow from node 1 to every other node along a shortest path.

The maximum flow problem. The maximum flow problem is to send the maximum possible flow in a network from a specified *source* node s to a specified *sink* node t. In the minimum cost flow problem, if we add an additional arc (t, s) with $c_{ts} = -1$ and $u_{ts} = \infty$, set the supply/demand of all nodes and costs of all arcs to zero, then the minimum cost solution maximizes the flow on arc (t, s) which equals the maximum possible flow from the source node to the sink node.

The assignment problem. The data of the assignment problem consists of a set N_1, say of persons, and a set N_2, say of objects, satisfying $|N_1| = |N_2|$, a collection of node pairs

$A \subseteq N_1 \times N_2$ representing possible person-to-object assignments, and a cost c_{ij} associated with each element (i, j) in A. The objective is to assign each person to exactly one object in a way that minimizes the cost of the assignment. The assignment problem is a minimum cost flow problem on a network $G = (N_1 \cup N_2, A)$ with $b(i) = 1$ for all $i \in N_1$ and $b(i) = -1$ for all $i \in N_2$ (we set $l_{ij} = 0$ and $u_{ij} = 1$ for all $(i, j) \in A$).

Physical Networks

The familiar city street map is perhaps the prototypical physical network, and the one that most readily comes to mind when we envision a network. Many network planning problems arise in this problem context. As one illustration, consider the problem of managing, or designing, a street network to decide upon such issues as speed limits, one way street assignments, or whether or not to construct a new road or bridge. In order to make these decisions intelligently, we need a descriptive model that tells us how to model traffic flows and measure the performance of any design as well as a predictive model for measuring the effect of any change in the system. We can then use these models to answer a variety of "what if" planning questions.

The following type of equilibrium network flow model permits us to answer these types of questions. Each line of the network has an associated delay function that specifies how long it takes to traverse this link. The time to do so depends upon traffic conditions; the more traffic that flows on the link, the longer is the travel time to traverse it. Now also suppose that each user of the system has a point of origin (e.g., his or her home) and a point of destination (e.g., his or her workplace in the central business district). Each of these users must choose a route through the network. Note, however, that these route choices affect each other; if two users traverse the same link, they add to each other's travel time because of the added congestion on the link. Now let us make the behavioral assumption that each user wishes to travel between his or her origin and destination as quickly as possible, that is, along a shortest travel time path. This situation leads to the following equilibrium problem with an embedded set of network optimization problems (shortest path problems); is there a flow pattern in the network with the property that no user can unilaterally change his (or her) choice of origin to destination path (that is, all other users continue to use their specified paths in the equilibrium solution) to reduce his travel time. Operations researchers have developed a set of sophisticated models for this problem setting, as well as related theory (concerning, for example, existence and uniqueness of equilibrium solutions), and algorithms for computing equilibrium solutions. Used in the mode of "what if"

scenario analysis, these models permit analysts to answer the type of questions we posed previously. These models are actively used in practice. Indeed, the Urban Mass Transit Authority in the United States requires that communities perform a network equilibrium impact analysis as part of the process for obtaining federal funds for highway construction or improvement.

Similar types of models arise in many other problem contexts. For example, a network equilibrium model forms the heart of the Project Independence Energy Systems (LPIES) model developed by the U.S. Department of Energy as an analysis tool for guiding public policy on energy. The basic equilibrium model of electrical networks is another example. In this setting, Ohm's Law serves as the analog of the congestion function for the traffic equilibrium problem, and Kirkhoff's Law represents the network mass balance equations.

Another type of physical network is a very large-scale integrated circuit (VLSI circuit). In this setting the nodes of the network correspond to electrical components and the links correspond to wires that connect these links. Numerous network planning problems arise in this problem context. For example, how can we lay out , or design, the smallest possible integrated circuit to make the necessary connections between its components and maintain necessary separations between the wires (to avoid electrical interference).

Route Networks

Route networks, which are one level of abstraction removed from physical networks, are familiar to most students of operations research and management science. The traditional operations research transportation problem is illustrative. A shipper with supplies at its plants must ship to geographically dispersed retail centers, each with a given customer demand. Each arc connecting a supply point to a retail center incurs costs based upon some physical network, in this case the transportation network. Rather than solving the problem directly on the physical network, we preprocess the data and construct transportation routes. Consequently, an arc connecting a supply point and retail center might correspond to a complex four leg distribution channel with legs (i) from a plant (by truck) to a rail station, (ii) from the rail station to a rail head elsewhere in the system, (iii) from the rail head (by truck) to a distribution center, and (iv) from the distribution center (on a local delivery truck) to the final customer (or even in some cases just to the distribution center). If we assign the arc with the composite

distribution cost of all the intermediary legs, as well as with the distribution capacity for this route, this problem becomes a classic network transportation model: find the flows from plants to customers that minimizes overall costs. This type of model is used in numerous applications. As but one illustration, a prize winning practice paper written several years ago described an application of such a network planning system by the Cahill May Roberts Pharmaceutical Company (of Ireland) to reduce overall distribution costs by 20%, while improving customer service as well.

Many related problems arise in this type of problem setting, for instance, the design issue of deciding upon the location of the distribution centers. It is possible to address this type of decision problem using integer programming methodology for choosing the distribution sites and network flows to cost out (or optimize flows) for any given choice of sites; using this approach, a noted study conducted several years ago permitted Hunt Wesson Foods Corporation to save over $1 million annually.

One special case of the transportation problem merits note -- the assignment problem that we introduced previously in this section. This problem has numerous applications, particularly in problem contexts such as machine scheduling. In this application context, we would identify the supply points with jobs to be performed, the demand points with available machines, and the cost associated with arc (i, j) as the cost of completing job i on machine j. The solution to the problem specifies the minimum cost assignment of the jobs to the machines, assuming that each machine has the capacity to perform only one job.

Space Time Networks

Frequently in practice, we wish to schedule some production or service activity over time. In these instances it is often convenient to formulate a network flow problem on a "space--time network" with several nodes representing a particular facility (a machine, a warehouse, an airport) but at different points in time.

Figure 1.2, which represents a core planning model in production planning, *the economic lot size problem*, is an important example. In this problem context, we wish to meet prescribed demands d_t for a product in each of the T time periods. In each period, we can produce at level x_t and/or we can meet the demand by drawing upon inventory I_t from the previous period. The network representing this problem has $T + 1$ nodes: one node $t = 1, 2, \ldots, T$ represents each of the planning periods, and one

node 0 represents the "source" of all production. The flow on arc (0, t) prescribes the production level x_t in period t, and the flow on arc (t, t + 1) represents the inventory level I_t to be carried from period t to period t + 1. The mass balance equation for each period t models the basic accounting equation: incoming inventory plus production in that period must equal demand plus final inventory. The mass balance equation for node 0 indicates that all demand (assuming zero beginning and zero final inventory over the entire planning period) must be produced in some period t = 1, 2, . . . , T. Whenever the production and holding costs are linear, this problem is easily solved as a shortest path problem (for each demand period, we must find the minimum cost path of production and inventory arcs from node 0 to that demand point). If we impose capacities on production or inventory, the problem becomes a minimum cost network flow problem.

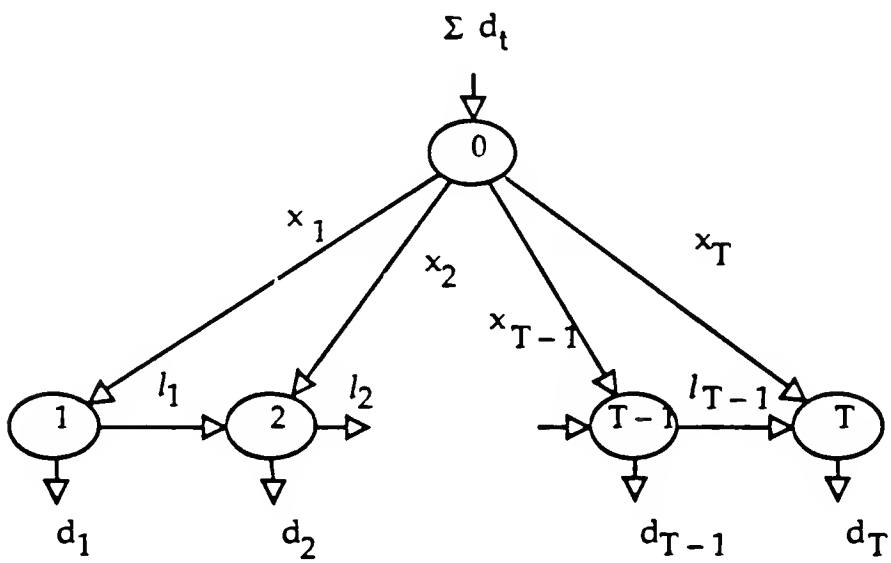


Figure 1.2. Network flow model of the economic lot size problem.

One extension of this economic lot sizing problem arises frequently in practice. Assume that production x_t in any period incurs a fixed cost: that is, whenever we produce in period t (i.e., $x_t > 0$), no matter how much or how little, we incur a fixed cost F_t. In addition , we may incur a per unit production cost c_t in period t and a per unit inventory cost h_t for carrying any unit of inventory from period t to period t + 1. Hence, the cost on each arc for this problem is either linear (for inventory carrying arcs) or linear plus a fixed cost (for production arcs). Consequently, the objective function for

the problem is concave. As we indicate in Section 2.2 , any such concave cost network flow problem always has a special type of optimum solution known as a spanning trees solution. This problem's spanning tree solution decomposes into disjoint directed paths; the first arc on each path is a production arc (of the form (0, t)) and each other arc is an inventory carrying arc. This observation implies the following *production property:* in the solution, each time we produce, we produce enough to meet the demand for an integral number of contiguous periods. Moreover, in no period do we both carry inventory from the previous period and produce.

The production property permits us to solve the problem very efficiently as a shortest path problem on an auxiliary network G' defined as follows. The network G' consists of nodes 1 to $T + 1$, and for every pair of nodes i and j with $i < j$, it contains an arc (i, j). The length of arc (i, j) is equal to the production and inventory cost of satisfying the demand of the periods from i to $j - 1$. Observe that for every production schedule satisfying the production property, G' contains a directed path in G' from node 1 to node $T + 1$ of the same objective function value and vice-versa. Hence we can obtain the optimum production schedule by solving a shortest path problem.

Many enhancements of the model are possible, for example (i) the production facility might have limited production capacity or limited storage for inventory, or (ii) the production facility might be producing several products that are linked by common production costs or by changeover cost (for example, we may need to change dies in an automobile stamping plant when making different types of fenders), or that share common limited production facilities. In most cases, the enhanced models are quite difficult to solve (they are NP-complete), though the embedded network structure often proves to be useful in designing either heuristic or optimization methods.

Another classical network flow scheduling problem is the *airline scheduling problem* used to identify a flight schedule for an airline. In this application setting, each node represents both a geographical location (e.g., an airport) and a point in time (e.g., New York at 10 A.M.). The arcs are of two types: (i) service arcs connecting two airports, for example New York at 10 A.M. to Boston at 11 A.M.; (ii) layover arcs that permit a plane to stay at New York from 10 A.M. until 11 A.M. to wait for a later flight, or to wait overnight at New York from 11 P.M. until 6 A.M. the next morning. If we identify revenues with each service leg, a network flow in this network (with no external supply or demand) will specify a set of flight plans (circulation of airplanes through the network). A flow that maximizes revenue will prescribe a schedule for an airline's fleet

of planes. The same type of network representation arises in many other dynamic scheduling applications.

Derived Networks

This category is a "grab bag" of specialized applications and illustrates that sometimes network flow problems arise in surprising ways from problems that on the surface might not appear to involve networks. The following examples illustrate this point.

Single Duty Crew Scheduling. Figure 1.3 illustrates a number of possible duties for the drivers of a bus company.

Time Period/Duty Number	1	2	3	4	5	6	7	8	9	10	11
9 – 10 A.M.	1	0	0	0	1	0	1	0	0	0	0
10 – 11 A.M.	0	0	0	0	1	0	1	0	0	0	0
11 – Noon	0	0	1	0	0	0	0	0	1	0	0
12 – 1 P.M.	0	0	1	0	0	0	0	0	1	0	0
1 – 2 P.M.	0	0	1	0	0	1	0	0	0	0	1
2 – 3 P.M.	0	1	0	0	0	1	0	1	0	0	1
3 – 4 P.M.	0	1	0	0	0	0	0	1	0	1	0
4 – 5 P.M.	0	0	0	1	0	0	0	0	0	1	0

Figure 1.3. Available duties for a single duty scheduling problem.

For example, the first duty (the first column in the table) represents a schedule in which a driver works form 9 A.M. to 10 A.M.; the second duty specifies that a driver works from 2 P.M. to 4 P.M. Suppose each duty j has an associated cost c_j. If we wish to ensure that a driver is on duty for each hour of the planning period (9 A.M. to 5 P.M. in the example), and the cost of scheduling is minimum, then the problem is an integer program:

$$\text{minimize} \quad cx \tag{1.2a}$$

$$\text{subject to} \quad Ax = b, \tag{1.2b}$$

$$x_j = 0 \text{ or } 1 \text{ for all } j. \tag{1.2c}$$

In this formulation the binary variable x_j indicates whether ($x_j = 1$) or not ($x_j = 0$) we select the j-th duty; the matrix A represents the matrix of duties and b is a column vector whose components are all 1's. Observe that the ones in each column of A occur in consecutive rows because each driver 's duty contains a single work shift (no split shifts or work breaks). We show that this problem is a shortest path problem. To make this identification, we perform the following operations: In (1.2b) subtract each equation from the equation below it. This transformation does not change the solution to the system. Now add a redundant equation equal to minus the sums of all the equations in the revised system. Because of the structure of A, each column in the revised system will have a single +1 (corresponding to the first hour of the duty in the column of A) and a single −1 (corresponding to the row in A, or the added row, that lies just below the last +1 in the column of A). Moreover, the revised right hand side vector of the problem will have a +1 in row 1 and a −1 in the last (the appended) row. Therefore, the problem is to ship one unit of flow from node 1 to node 9 at minimum cost in the network given in Figure 1.4, which is an instance of the shortest path problem.

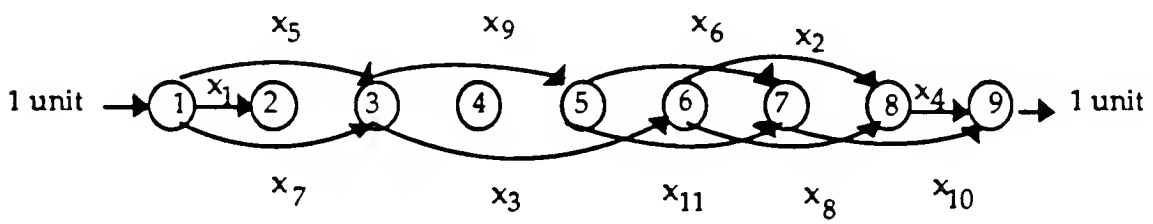


Figure 1.4. Shortest path formulation of the single duty scheduling problem.

If instead of requiring a single driver to be on duty in each period, we specify a number to be on duty in each period, the same transformation would produce a network flow problem, but in this case the right hand side coefficients (supply and demands) could be arbitrary. Therefore, the transformed problem would be a general minimum cost network flow problem, rather than a shortest path problem.

Critical Path Scheduling and Networks Derived from Precedence Conditions

In construction and many other project planning applications, workers need to complete a variety of tasks that are related by precedence conditions; for example, in constructing a house, a builder must pour the foundation before framing the house and complete the framing before beginning to install either electrical or plumbing fixtures.

This type of application can be formulated mathematically as follows. Suppose we need to complete J jobs and that job j (j = 1, 2, . . . , J) requires t_j days to complete. We are to choose the start time s_j of each job j so that we honor a set of specified precedence constraints and complete the overall project as quickly as possible. If we represent the jobs by nodes, then the precedence constraints can be represented by arcs, thereby giving us a network. The precedence constraints imply that for each arc (i, j) in the network, the job j cannot start until job i has been completed. For convenience of notation, we add two dummy jobs, both with zero processing time: a "start" job 0 to be completed before any other job can begin and a "completion" job J + 1 that cannot be initiated until we have completed all other jobs. Let G = (N, A) represent the network corresponding to this augmented project. Then we wish to solve the following optimization problem:

$$\text{minimize } s_{J+1} - s_0 ,$$

subject to

$$s_j \geq s_i + t_i , \text{ for each arc } (i , j) \in A.$$

On the surface, this problem, which is a linear program in the variables s_j, seems to bear no resemblance to network optimization. Note, however, that if we move the variable s_i to the left hand side of the constraint, then each constraint contains exactly two variables, one with a plus one coefficient and one with a minus one coefficient. The linear programming dual of this problem has a familiar structure. If we associate a dual variable x_{ij} with each arc (i, j), then the dual of this problem is

$$\text{maximize } \sum_{(i, j) \in A} t_i x_{ij} ,$$

subject to

$$-\sum_{\{j:\,(i, j) \in A\}} x_{ij} + \sum_{\{j:\,(j, i) \in A\}} x_{ji} = \begin{cases} -1, \text{ if } i = 0 \\ 0, \text{ otherwise, for all } i \in N \\ 1, \text{ if } i = J + 1 \end{cases}$$

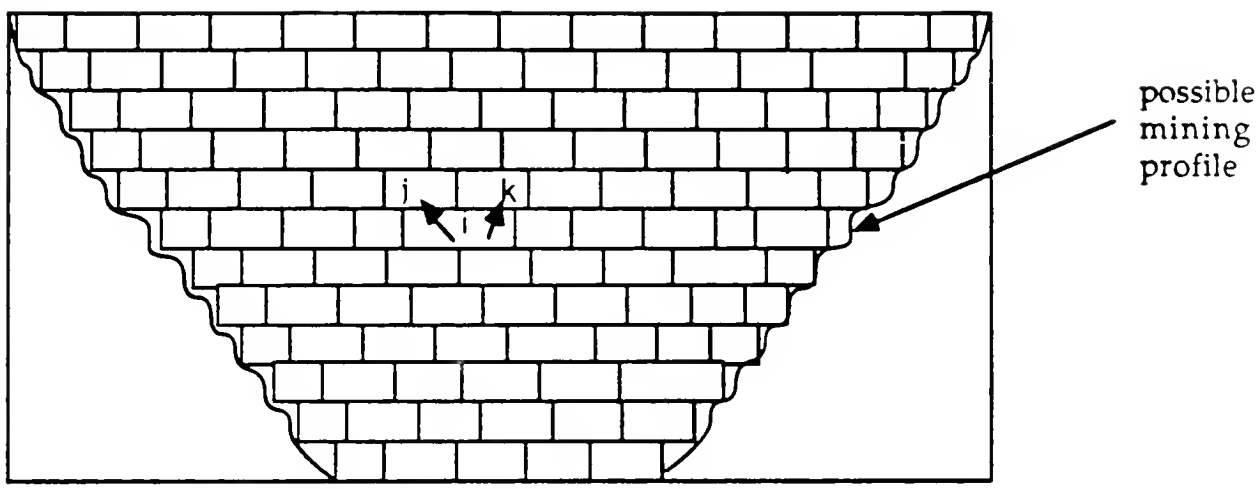


Figure 1.5. Open pit mine; we must extract blocks j and k before block i.

$$x_{ij} \geq 0, \text{ for all } (i, j) \in A .$$

This problem requires us to determine the longest path in the network G from node 0 to node J + 1 with t_i as the arc length of arc (i, j). This longest path has the following interpretation. It is the longest sequence of jobs needed to fulfill the specified precedence conditions. Since delaying any job in this sequence must necessarily delay the completion of the overall project, this path has become known as the *critical path* and the problem has become known as the *critical path problem*. This model has become a principal tool in project management, particularly for managing large-scale construction projects. The critical path itself is important because it identifies those jobs that require managerial attention in order to complete the project as quickly as possible.

Researchers and practitioners have enhanced this basic model in several ways. For example, if resources are available for expediting individual jobs, we could consider the most efficient use of these resources to complete the overall project as quickly as possible. Certain versions of this problem can be formulated as minimum cost flow problems.

The *open pit mining* problem is another network flow problem that arises from precedence conditions. Consider the open pit mine shown in Figure 1.5. As shown in this figure, we have divided the region to be mined into blocks. The provisions of any given mining technology, and perhaps the geography of the mine, impose restrictions on how we can remove the blocks: for example, we can never remove a block until we have removed any block that lies immediately above it; restrictions on the "angle" of mining the blocks might impose similar precedence conditions. Suppose now that each block j has an associated revenue r_j (e.g., the value of the ore in the block minus the cost for extracting the block) and we wish to extract blocks to maximize overall revenue. If we let y_j be a zero-one variable indicating whether ($y_i = 1$) or not ($y_j = 0$) we extract block j, the problem will contain (i) a constraint $y_j \leq y_i$ (or, $y_j - y_i \leq 0$) whenever we need to mine block j before block i, and (ii) an objective function specifying that we wish to maximize total revenue $r_j y_j$, summed over all blocks j. The dual linear program (obtained from the linear programming version of the problem (with the constraints $0 \leq y_j \leq 1$, rather than $y_j = 0$ or 1) will be a network flow problem with a node for each block, a variable for each precedence constraint, and the revenue r_j as the demand at node j. This network will also have a dummy "collection node" 0 with demand equal to minus the sum of the r_j's, and an arc connecting it to node j (that is,

block j); this arc corresponds to the upper bound constraint $y_j \leq 1$ in the original linear program. The dual problem is one of finding a network flow that minimizes ths sum of flows on the arcs incident to node 0.

The critical path scheduling problem and open pit mining problem illustrate one way that network flow problems arise indirectly. Whenever, two variables in a linear program are related by a precedence conditions, the variable corresponding to this precedence constraint in the dual linear program will have a network flow structure. If the only constraints in the problem are precedence constraints, the dual linear program will be a network flow problem.

Matrix Rounding of Census Information

The U.S. Census Bureau uses census information to construct millions of tables for a wide variety of purposes. By law, the Bureau has an obligation to protect the source of its information and not disclose statistics that can be attributed to any particular individual. It can attempt to do so by rounding the census information contained in any table. Consider, for example, the data shown in Figure 1.6(a). Since the upper leftmost entry in this table is a 1, the tabulated information might disclose information about a particular individual. We might disguise the information in this table as follows; round each entry in the table, including the row and column sums, either up or down to the nearest multiple of three, say, so that the entries in the table continue to add to the (rounded) row and column sums, and the overall sum of the entries in the new table adds to a rounded version of the overall sum in the original table. Figure 1.6(b) shows a rounded version of the data that meets this criterion. The problem can be cast as finding a feasible flow in a network and can be solved by an application of the maximum flow algorithm. The network contains a node for each row in the table and one node for each column. It contains an arc connecting node i (corresponding to row i) and node j (corresponding to column j): the flow on this arc should be the ij-th entry in the prescribed table, rounded either up or down. In addition, we add a supersource s to the network connected to each row node i: the flow on this arc must be the i-th row sum, rounded up or down. Similarly, we add a supersink t with the arc connecting each column node j to this node; the flow on this arc must be the j-th column sum, rounded up or down. We also add an arc connecting node t and node s; the flow on this arc must be the sum of all entries in the original table rounded up or down. Figure 1.7 illustrates the network flow problem corresponding to the census data specified in Figure 1.6. If we rescale all the flows, measuring them in integral units of the rounding base

Income	Time in service (hours) < 1	1 – 5	< 5	Row Total
less than $10,000	1	5	1	7
$10,000 – $30,000	0	2	2	4
$30,000 – $50,000	0	2	1	3
more than $50,000	0	1	4	5
Column Total	1	10	8	19

(a) Raw Data

Income	Time in service (hours) < 1	1 – 5	> 5	Row Total
less than $10,000	3	3	0	6
$10,000 – $30,000	0	3	3	6
$30,000 – $50,000	0	3	0	3
more than $50,000	0	0	6	6
Column Total	3	9	9	21

(b) Rounded Data

Figure 1.6. Rounding Census Table

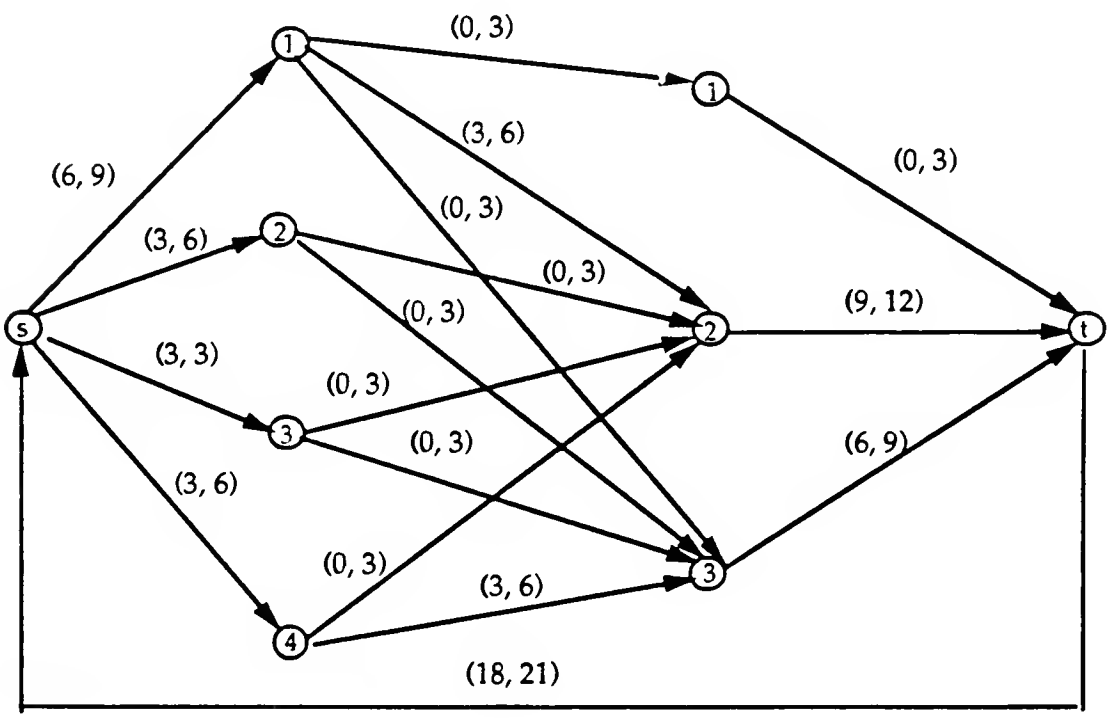


Figure 1.7. Network flow problem for census rounding: the network shows lower and upper bounds for arc flows.

(multiples of 3 in our example), then the flow on each arc must be integral at one of two consecutive integral values. The formulation of a more general version of this problem, corresponding to tables with more than two dimensions, will not be a network flow problem. Nevertheless, these problems have an imbedded network structure (corresponding to 2-dimensional "cuts" in the table) that we can exploit in divising algorithms to find rounded versions of the tables.

1.2 Complexity Analysis

There are three basic approaches for measuring the performance of an algorithm: empirical analysis, worst-case analysis, and average-case analysis. *Empirical analysis* typically measures the computational time of an algorithm using statistical sampling on a distribution (or several distributions) of problem instances. The major objective of empirical analysis is to estimate how algorithms behave in practice. *Worst-case analysis* aims to provide upper bounds on the number of steps that a given algorithm can take on *any* problem instance. Therefore, this type of analysis provides *performance guarantees*. The objective of *average-case analysis* is to estimate the expected number of steps taken by an algorithm. Average-case analysis differs from empirical analysis because it provides rigorous mathematical proofs of average-case performance, rather than statistical estimates.

Each of these three performance measures has its relative merits, and is appropriate for certain purposes. Nevertheless, this chapter will focus primarily on worst-case analysis, and only secondarily on empirical behavior. Researchers have designed many of the algorithms described in this chapter specifically to improve worst-case complexity while simultaneously maintaining good empirical behavior. Thus, for the algorithms we present, worst-case analysis is the primary measure of performance.

Worst-Case Analysis

For worst-case analysis, we bound the running time of network algorithms in terms of several basic problem parameters: the number of nodes (n), the number of arcs (m), and upper bounds C and U on the cost coefficients and the arc capacities. Whenever C (or U) appears in the complexity analysis, we assume that each cost (or capacity) is integer valued. As an example of a worst-case result within this chapter, we will prove

that the number of steps for the label correcting algorithm to solve the shortest path problem is less than pnm steps for some sufficiently large constant p.

To avoid the need to compute or mention the constant p, researchers typically use a "big O" notation, replacing the expressions: "the label correcting algorithm requires pmn steps for some constant p" with the equivalent expression "the running time of the label correcting algorithm is O(nm)." The O( ) notation avoids the need to state a specific constant; instead, this notation indicates only the *dominant* terms of the running time. By dominant, we mean the term that would dominate all other terms for *sufficiently* large values of n and m. Therefore, the time bounds are called *asymptotic* running times. For example, if the actual running time is $10nm^2 + 2^{100}n^2m$, then we would state that the running time is $O(nm^2)$, assuming that $m \geq n$. Observe that the running time indicates that the $10nm^2$ term is dominant even though for most practical values of n and m, the $2^{100}n^2m$ term would dominate. Although ignoring the constant terms may have this undesirable feature, researchers have widely adopted the O( ) notation for several reasons:

1. Ignoring the constants greatly simplifies the analysis. Consequently, the use of the O( ) notation typically has permited analysts to avoid the prohibitively difficult analysis required to compute the leading constants, which, in turn, has led to a flourishing of research on the worst-case performance of algorithms.

2. Estimating the constants correctly is fundamentally difficult. The least value of the constants is not determined solely by the algorithm; it is also highly sensitive to the choice of the computer language, and even to the choice of the computer.

3. For all of the algorithms that we present, the constant terms are relatively small integers for all the terms in the complexity bound.

4. For large practical problems, the constant factors do not contribute nearly as much to the running time as do the factors involving n, m, C or U.

Counting Steps

The running time of a network algorithm is determined by counting the number of steps it performs. The counting of steps relies on a number of assumptions, most of which are quite appropriate for most of today's computers.

A1.1 The computer carries out instructions sequentially, with at most one instruction being executed at a time.

A1.2 Each comparison and basic arithmetic operation counts as one step.

By envoking A1.1, we are adhering to a sequential model of computations; we will not discuss parallel implementations of network flow algorithms.

A1.2 implicitly assumes that the only operations to be counted are comparisons and arithmetic operations. In fact, even by counting all other computer operations, on today's computers we would obtain the same asymptotic worst-case results for the algorithms that we present. Our assumption that each operation, be it an addition or division, takes equal time, is justified in part by the fact that O( ) notation ignores differences in running times of at most a constant factor, which is the time difference between an addition and a multiplication on essentially all modern computers.

On the other hand, the assumption that each arithmetic operation takes one step may lead us to underestimate the asymptotic running time of arithmetic operations involving very large numbers on real computers since, in practice, a computer must store large numbers in several words of its memory. Therefore, to perform each operation on very large numbers, a computer must access a number of words of data and thus takes more than a constant number of steps. To avoid this systematic underestimation of the running time, in comparing two running times, we will typically assume that both C and U are polynomially bounded in n, i.e., $C = O(n^k)$ and $U = O(n^k)$, for some constant k. This assumption, known as the *similarity assumption*, is quite reasonable in practice. For example, if we were to restrict costs to be less than $100n^3$, we would allow costs to be as large as 100,000,000,000 for networks with 1000 nodes.

Polynomial-Time Algorithms

An algorithm is said to be a *polynomial-time* algorithm if its running time is bounded by a polynomial function of the input length. The *input length* of a problem is the number of bits needed to represent that problem. For a network problem, the input length is a low order polynomial function of n, m, log C and log U (e.g., it is $O((n + m)(\log n + \log C + \log U))$. Consequently, researchers refer to a network algorithm as a polynomial-time algorithm if its running time is bounded by a polynomial function in n, m, log C and log U. For example, the running time of one of the polynomial-time maximum flow algorithms we consider is $O(nm + n^2 \log U)$. Other instances of

polynomial-time bounds are $O(n^2m)$ and $O(n \log n)$. A polynomial-time algorithm is said to be a *strongly polynomial-time* algorithm if its running time is bounded by a polynomial function in only n and m, and does not involve log C or log U. The maximum flow algorithm alluded to, therefore, is not a strongly polynomial-time algorithm. The interest in strongly polynomial-time algorithms is primarily theoretical. In particular, if we envoke the similarity assumption, all polynomial-time algorithms are strongly polynomial-time because $\log C = O(\log n)$ and $\log U = O(\log n)$.

An algorithm is said to be an *exponential-time* algorithm if its running time grows as a function that can not be polynomially bounded. Some examples of exponential time bounds are $O(nC)$, $O(2^n)$, $O(n!)$ and $O(n^{\log n})$. (Observe that nC cannot be bounded by a polynomial function of n and log C.) We say that an algorithm is *pseudopolynomial-time* if its running time is polynomially bounded in n, m, C and U. The class of pseudopolynomial-time algorithms is an important subclass of exponential-time algorithms. Some instances of pseudopolynomial-time bounds are $O(m + nC)$ and $O(mC)$. For problems that satisfy the similarity assumption, pseudopolynomial-time algorithms become polynomial-time algorithms, but the algorithms will not be attractive if C and U are high degree polynomials in n.

There are two major reasons for preferring polynomial-time algorithms to exponential-time algorithms. First, any polynomial-time algorithm is asymptotically superior to any exponential-time algorithm. Even in extreme cases this is true. For example, $n^{1,000}$ is smaller than $n^{0.1\log n}$ if n is sufficiently large. (In this case, n must be larger than $2^{100,000}$.) Figure 1.8 illustrates the asymptotic superiority of polynomial-time algorithms. The second reason is more pragmatic. Much practical experience has shown that, as a rule, polynomial-time algorithms perform better than exponential time algorithms. Moreover, the polynomials in practice are typically of a small degree.

APPROXIMATE VALUES							
n	log n	$n^{.5}$	n^2	n^3	$n^{\log n}$	2^n	n!
10	3.32	3.16	10^2	10^3	2.10×10^3	10^3	3.6×10^6
100	6.64	10.00	10^4	10^6	1.94×10^{13}	1.27×10^{30}	9.33×10^{157}
1000	9.97	31.62	10^6	10^9	7.90×10^{29}	1.07×10^{301}	$4.02 \times 10^{2,567}$
10,000	13.29	100.00	10^8	10^{12}	1.42×10^{53}	$.99 \times 10^{3,010}$	$2.85 \times 10^{35,659}$

Figure 1.8. The growth of polynomial and exponential functions.

In computational complexity theory, the basic objective is to obtain polynomial-time algorithms, preferably ones with the lowest possible degree. For example, O(log n) is preferable to $O(n^k)$ for any $k > 0$, and $O(n^2)$ is preferable to $O(n^3)$. However, running times involving more than one parameter, such as O(nm log n) and $O(n^3)$, may not be comparable. If $m < n^2/\log n$ then O(nm log n) is superior; otherwise $O(n^3)$ is superior.

Related to the O( ) notation is the Ω( ), or "big omega", notation. Just as O( ) specifies an upper bound on the computational time of an algorithm, to within a constant factor, Ω( ) specifies a lower bound on the computational time of an algorithm, again to within a constant factor. We say that an algorithm runs in Ω(f(n, m)) time if for some example the algorithm can indeed take q f(n, m) time for some constant q. For example, it is possible to show that the label correcting algorithm that we consider in Section 3.4 for the shortest path problem can take qnm time. Therefore, we write the equivalent statement, "the running time of the label correcting algorithm is Ω(nm)."

1.3 Notation and Definitions

For convenience, in this section we collect together several basic definitions and describe some basic notation. We also state without proof some elementary properties of graphs.

We consider a *directed graph* G = (N, A) consisting of a set, N, of nodes, and a set, A, of arcs whose elements are ordered pairs of distinct nodes. A *directed network* is a directed graph with numerical values attached to its nodes and/or arcs. As before, we

$|N|$ and $m = |A|$. We associate with each arc $(i, j) \in A$, a cost c_{ij} and a capacity u_{ij}. We assume throughout that $u_{ij} \geq 0$ for each $(i, j) \in A$. Frequently, we distinguish two special nodes in a graph: the *source* s and *sink* t.

An arc (i, j) has two *end points*, i and j. The arc (i,j) is incident to nodes i and j. We refer to node i as the *tail* and node j as the *head* of arc (i, j), and say that the arc (i, j) *emanates from* node i. The arc (i, j) is an *outgoing* arc of node i and an *incoming* arc of node j. The *arc adjacency list* of node i, A(i), is defined as the set of arcs emanating from node i, i.e., $A(i) = \{(i, j) \in A : j \in N\}$. The degree of a node is the number of incoming and outgoing arcs incident to that node.

A *directed path* in G = (N, A) is a sequence of distinct nodes and arcs $i_1, (i_1, i_2), i_2, (i_2, i_3), i_3, \ldots, (i_{r-1}, i_r), i_r$ satisfying the property that $(i_k, i_{k+1}) \in A$ for each $k = 1, \ldots, r-1$. An *undirected path* is defined similarly except that for any two consecutive nodes i_k and i_{k+1} on the path, the path contains either arc (i_k, i_{k+1}) or arc (i_{k+1}, i_k). We refer to the nodes $i_2, i_3, \ldots, i_{r-1}$ as the *internal* nodes of the path. A *directed cycle* is a directed path together with the arc (i_r, i_1) and an undirected cycle is an undirected path together with the arc (i_r, i) or (i_1, i_r).

We shall often use the terminology *path* to designate either a directed or an undirected path; whichever is appropriate from context. If any ambiguity might arise, we shall explicitly state directed or undirected path. For simplicity of notation, we shall often refer to a path as a sequence of nodes $i_1 - i_2 - \ldots - i_k$ when its arcs are apparent from the problem context. Alternatively, we shall sometimes refer to a path as a set of (sequence of) arcs without mention of the nodes. We shall use similar conventions for representing cycles.

A graph G = (N, A) is called a *bipartite graph* if its node set N can be partitioned into two subsets N_1 and N_2 so that for each arc (i, j) in A, $i \in N_1$ and $j \in N_2$.

A graph G' = (N', A') is a *subgraph* of G = (N, A) if $N' \subseteq N$ and $A' \subseteq A$. A graph G' = (N', A') is a spanning subgraph of G = (N, A) if N' = N and $A' \subseteq A$.

Two nodes i and j are said to be *connected* if the graph contains at least one undirected path from i to j. A graph is said to be *connected* if all pairs of nodes are connected; otherwise, it is *disconnected*. In this chapter, we always assume that the graph G is connected. We refer to any set $Q \subseteq A$ with the property that the graph G' = (N, A-Q) is disconnected, and no superset of Q has this property, as a *cutset* of G. A cutset

partitions the graph into two sets of nodes, X and N-X. We shall alternatively represent the cutset Q as the node partition (X, N-X).

A graph is *acyclic* if it contains no cycle. A *tree* is a connected acyclic graph. A *subtree* of a tree T is a connected subgraph of T. A tree T is said to be a *spanning tree* of G if T is a spanning subgraph of G. Arcs belonging to a spanning tree T are called *tree arcs*, and arcs not belonging to T are called *nontree arcs*. A spanning tree of G = (N, A) has exactly n-1 tree arcs. A node in a tree with degree equal to one is called a *leaf* node. Each tree has at least two leaf nodes.

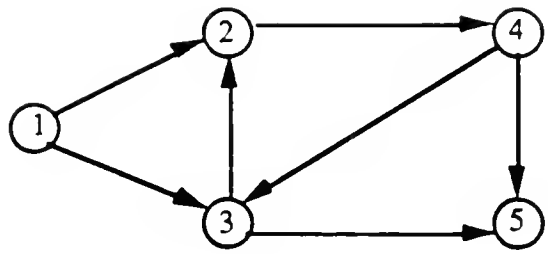
A spanning tree contains a unique path between any two nodes. The addition of any nontree arc to a spanning tree creates exactly one cycle. Removing any arc in this cycle again creates a spanning tree. Removing any tree-arc creates two subtrees. Arcs whose end points belong to two different subtrees of a spanning tree created by deleting a tree-arc constitute a cutset. If any arc belonging to this cutset is added to the subtrees, the resulting graph is again a spanning tree.

In this chapter, we assume that logarithms are of base 2 unless we state it otherwise. We represent the logarithm of any number b by log b.

1.4 Network Representations

The complexity of a network algorithm depends not only on the algorithm, but also upon the manner used to represent the network within a computer and the storage scheme used for maintaining and updating the intermediate results. The running time of an algorithm (either worst-case or empirical) can often be improved by representing the network more cleverly and by using improved data structures. In this section, we discuss some popular ways of representing a network.

In Section 1.1, we have already described the node-arc incidence matrix representation of a network. This scheme requires nm words to store a network, of which only 2m words have nonzero values. Clearly, this network representation is not space efficient. Another popular way to represent a network is the *node-node adjacency matrix representation*. This representation stores an $n \times n$ matrix I with the property that the element $I_{ij} = 1$ if arc $(i, j) \in A$, and $I_{ij} = 0$ otherwise. The arc costs and capacities are



(a) A network example

arc number	point	(tail, head)	cost
1	1	(1, 2)	2
2	3	(1, 3)	3
3	4	(2, 4)	1
4	6	(3, 2)	4
5	8	(3, 5)	2
6	9	(4, 5)	4
7		(4, 3)	1
8		(5, 4)	3

(b) The forward star representation.

cost	(tail, head)	rpoint
4	(3, 2)	1
2	(1, 2)	1
3	(1, 3)	3
1	(4, 3)	5
1	(2, 4)	7
3	(5, 4)	9
4	(4, 5)	
2	(3, 5)	

(c) The reverse star representation.

arc number	point	(tail, head)	cost	trace	rpoint
1	1	(1, 2)	2	4	1
2	3	(1, 3)	3	1	1
3	4	(2, 4)	1	2	3
4	6	(3, 2)	4	7	5
5	8	(3, 5)	2	3	7
6	9	(4, 5)	4	8	9
7		(4, 3)	1	6	
8		(5, 4)	3	5	

(d) The forward and reverse star representations.

Figure 1.9. Example of forward and reverse star network representations.

also stored in n × n matrices. This representation is adequate for very dense networks, but is not attractive for storing a sparse network.

The *forward star* and *reverse star representations* are probably the most popular ways to represent networks, both sparse and dense. (These representations are also known as *incidence list representation* in the computer science literature.) The forward star representation numbers the arcs in a certain order: we first number the arcs emanating from node 1, then the arcs emanating from node 2, and so on. Arcs emanating from the same node can be numbered arbitrarily. We then sequentially store the (tail, head) and the cost of arcs in this order. We also maintain a pointer with each node i, denoted by *point(i)*, that indicates the smallest number in the arc list of an arc emanating from node i. Hence the outgoing arcs of node i are stored at positions point(i) to (point(i+1) - 1) in the arc list. If point(i) > point(i+1) - 1, then node i has no outgoing arc. For consistency, set point(1) = 1 and point(n+1) = m+1. Figure 1.9(b) specifies the forward star representation of the network given in Figure 1.9(a).

The forward star representation allows us to determine efficiently the set of outgoing arcs at any node. To determine, simultaneously, the set of incoming arcs at any node efficiently, we need an additional data structure known as the *reverse star representation*. Starting from a forward star representation, we can create a reverse star representation as follows. We examine the nodes j = 1 to n in order and sequentially store the (tail, head) and the cost of incoming arcs of node j. We also maintain a reverse pointer with each node i, denoted by *rpoint(i)*, which denotes the first position in these arrays that contains information about an incoming arc at node i. For the sake of consistency, we set rpoint(1) = 1 and rpoint(n+1) = m+1. As earlier, we store the incoming arcs at node i at positions rpoint(i) to (rpoint(i+1) - 1). This data structure gives us the representation shown in Figure 1.9(c).

Observe that by storing both the forward and reverse star representation S, we will maintain a significant duplicate information. We can avoid this duplication by storing arc numbers instead of the (tail, head) and the cost of the arcs. For example, arc (3, 2) has arc number 4 in the forward star representation. The arc (1, 2) has arc number 1. So instead of storing (tail, head) and cost of arcs, we can simply store the arc numbers and once we know the arc numbers, we can always retrieve the associated information from the forward star representation. We store arc numbers in an m-array *trace*. Figure 1.9(d) gives the complete trace array.

1.5 Search Algorithms

Search algorithms are fundamental graph techniques; different variants of search lie at the heart of many network algorithms. In this section, we discuss two of the most commonly used search techniques: *breadth-first search* and *depth-first search*.

Search algorithms attempt to find all nodes in a network that satisfy a particular property. For purposes of illustration, let us suppose that we wish to find all the nodes in a graph G = (N, A) that are reachable through directed paths from a distinguished node s, called the *source*. At every point in the search procedure, all nodes in the network are in one of two states: *marked* or *unmarked*. The marked nodes are known to be reachable from the source, and the status of unmarked nodes is yet to be determined. We call an arc (i, j) *admissible* if node i is marked and node j is unmarked, and *inadmissible* otherwise. Initially, only the source node is marked. Subsequently, by examining admissible arcs, the search algorithm will mark more nodes. Whenever the procedure marks a new node j by examining an admissible arc (i, j) we say that node i is a *predecessor* of node j, i.e., *pred*(j) = i. The algorithm terminates when the graph contains no admissible arcs. The following algorithm summarizes the basic iterative steps.

```
algorithm SEARCH;
begin
        unmark all nodes in N;
        mark node s;
        LIST := {s};
        while LIST ≠ ø do
        begin
                select a node i in LIST;
                if node i is incident to an admissible arc (i, j) then
                begin
                        mark node j;
                        pred(j) := i;
                        add node j to LIST;
                end
                else delete node i from LIST;
        end;
end;
```

When this algoirthm terminates, it has marked all nodes in G that are reachable from s via a directed path. The predecessor indices define a tree consisting of marked nodes.

We use the following data structure to identify admissible arcs. The same data structure is also used in the maximum flow and minimum cost flow algorithms discussed in later sections. We maintain with each node i the list A(i) of arcs emanating from it. Arcs in each list can be arranged arbitrarily. Each node has a *current arc* (i, j) which is the current candidate for being examined next. Initially, the current arc of node i is the first arc in A(i). The search algorithm examines this list sequentially and whenever the current arc is inadmissible, it makes the next arc in the arc list the current arc. When the algorithm reaches the end of the arc list, it declares that the node has no admissible arc.

It is easy to show that the search algorithm runs in O(m + n) = O(m) time. Each iteration of the **while** loop either finds an admissible arc or does not. In the former case, the algorithm marks a new node and adds it to LIST, and in the latter case it deletes a marked node from LIST. Since the algorithm marks any node at most once, it executes the **while** loop at most 2n times. Now consider the effort spent in identifying the

admissible arcs. For each node i, we scan arcs in A(i) at most once. Therefore, the search algorithm examines a total of $\sum_{i \in N} A(i) = m$ arcs, and thus terminates in O(m) time.

The algorithm, as described, does not specify the order for examining and adding nodes to LIST. Different rules give rise to different search techniques. If the set LIST is maintained as a *queue*, i.e., nodes are always selected from the front and added to the rear, then the search algorithm selects the marked nodes in the first-in, first-out order. This kind of search amounts to visiting the nodes in order of increasing distance from s; therefore, this version of search is called a *breadth-first search*. It marks nodes in the nondecreasing order of their distance from s, with the distance from s to i measured as the minimum number of arcs in a directed path from s to i.

Another popular method is to maintain the set LIST as a *stack*, i.e., nodes are always selected from the front and added to the front; in this instance, the search algorithm selects the marked nodes in the last-in, first-out order. This algorithm performs a deep probe, creating a path as long as possible, and backs up one node to initiate a new probe when it can mark no new nodes from the tip of the path. Hence, this version of search is called a *depth-first search*.

1.6 Developing Polynomial-Time Algorithms

Researchers frequently employ two important approaches to obtain polynomial algorithms for network flow problems: the *geometric improvement* (or linear convergence) approach, and the *scaling* approach. In this section, we briefly outline the basic ideas underlying these two approaches. We will assume, as usual, that all data are integral and that algorithms maintain integer solutions at intermediate stages of computations.

Geometric Improvement Approach

The geometric improvement approach shows that an algorithm runs in polynomial time if at every iteration it makes an improvement proportional to the difference between the objective function values of the current and optimum solutions. Let H be an upper bound on the difference in objective function values between any two feasible solutions. For most network problems, H is a function of n, m, C, and U. For instance, in the maximum flow problem H = mU, and in the minimum cost flow problem H = mCU.

Lemma 1.1. Suppose z^k *is the objective function value of a minimization problem of some solution at the k-th iteration of an algorithm and* z^* *is the minimum objective function value. Further, suppose that the algorithm guarantees that*

$$(z^k - z^{k+1}) \geq \alpha (z^k - z^*) \qquad (1.3)$$

(i.e., the improvement at iteration k+1 is at least α *times the total possible improvement) for some constant* α *with* $0 < \alpha < 1$. *Then the algorithm terminates in* $O((\log H)/\alpha)$ *iterations.*

Proof. The quantity $(z^k - z^*)$ represents the total possible improvement in the objective function value after the k-th iteration. Consider a consecutive sequence of $2/\alpha$ iterations starting from iteration k. If in each iteration, the algorithm improves the objective function value by at least $\alpha(z^k - z^*)/2$ units, then the algorithm would determine an optimum solution within these $2/\alpha$ iterations. On the other hand, if at some iteration, q the algorithm improves the objective function value by no more than $\alpha(z^k - z^*)/2$ units, then (1.3) implies that

$$\alpha(z^k - z^*)/2 \geq z^q - z^{q+1} \geq \alpha(z^q - z^*),$$

and, therefore, the algorithm must have reduced the total possible improvement $(z^k - z^*)$ by a factor of 2 within these $2/\alpha$ iterations. Since H is the maximum possible improvement and every objective function value is an integer, the algorithm must terminate within $O((\log H)/\alpha)$ iterations. ■

We have stated this result for minimization versions of optimization problems. A similar result applies to maximization versions of optimization problems.

The geometric improvement approach might be summarized by the statement "network algorithms that have a geometric convergence rate are polynomial time algorithms." In order to develop polynomial time algorithms using this approach, we can look for local improvement techniques that lead to large (i.e., fixed percentage) improvements in the objective function. The maximum augmenting path algorithm for the maximum flow problem and the maximum improvement algorithm for the minimum cost flow problem are two examples of this approach. (See Sections 4.2 and 5.3.)

Scaling Approach

Researchers have extensively used an approach called *scaling* to derive polynomial-time algorithms for a wide variety of network and combinatorial optimization problems. In this discussion, we describe the simplest form of scaling which we call *bit-scaling*. Section 5.11 presents an example of a bit-scaling algorithm for

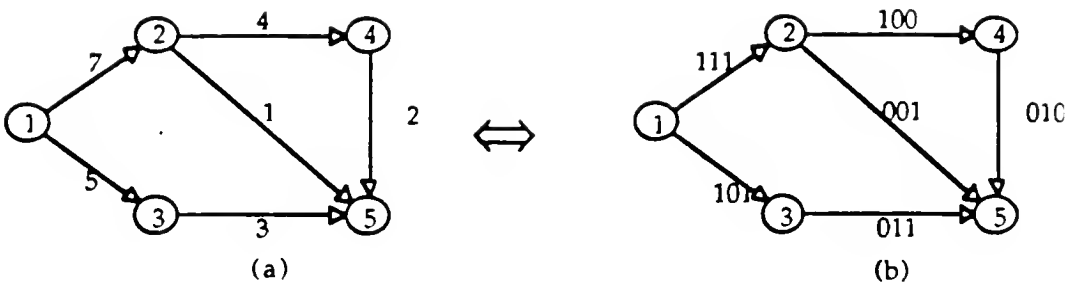
the assignment problem. Sections 4 and 5, using more refined versions of scaling, describe polynomial-time algorithms for the maximum flow and minimum cost flow problems.

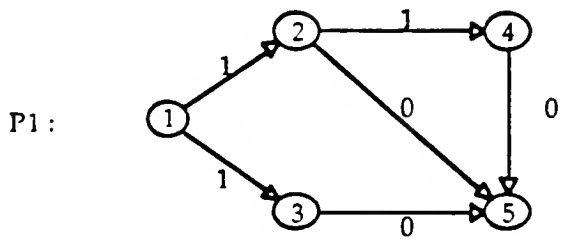
Using the bit-scaling technique, we solve a problem P parametrically as a sequence of problems $P_1, P_2, P_3, \ldots, P_K$: the problem P_1 approximates data to the first *bit*, the problem P_2 approximates data to the second bit, and each successive problem is a better approximation until $P_K = P$. Further, for each $k = 2, \ldots, K$, the optimum solution of problem P_{k-1} serves as the starting solution for problem P_k. The scaling technique is useful whenever reoptimization from a good starting solution is more efficient than solving the problem from scratch.

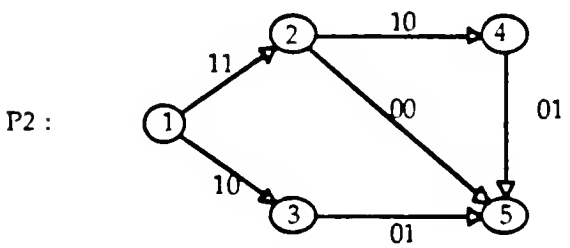
For example, consider a network flow problem whose largest arc capacity has value U. Let $K = \lceil \log U \rceil$ and suppose that we represent each arc capacity as a K bit binary number, adding leading zeros if necessary to make each capacity K bits long. Then the problem P_k would consider the capacity of each arc as the k leading bits in its binary representation. Figure 1.10 illustrates an example of this type of scaling.

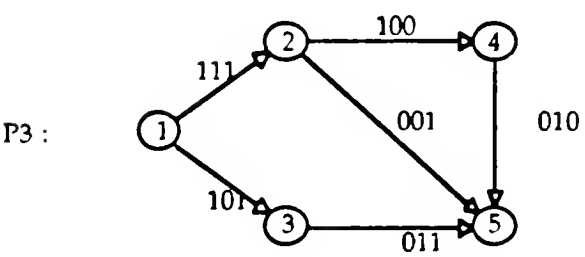
The manner of defining arc capacities easily implies the following observation.

Observation. The capacity of an arc in P_k is twice that in P_{k-1} plus 0 or 1.









(c)

Figure 1.10. Example of a bit-scaling technique.

(a) Network with arc capacities.

(b) Network with binary expansion of arc capacities.

(c) The problems P_1, P_2, and P_3.

The following algorithm encodes a generic version of the bit-scaling technique.

```
algorithm BIT-SCALING;
begin
    obtain an optimum solution of P_1;
    for k := 2 to K do
    begin
        reoptimize using the optimum solution of P_{k-1} to
            obtain an optimum solution of P_k;
    end;
end;
```

This approach is very robust; variants of it have led to improved algorithms for both the maximum flow and minimum cost flow problems. This approach works well for these applications, in part, because of the following reasons. (i) The problem P_1 is generally easy to solve. (ii) The optimal solution of problem P_{k-1} is an excellent starting solution for problem P_k since P_{k-1} and P_k are quite similar. Hence, the optimum solution of P_{k-1} can be easily reoptimized to obtain an optimum solution of P_k. (iii) For problems that satisfy the similarity assumption, the number of problems solved is O(log n). Thus for this approach to work, reoptimization needs to be only a little more efficient (i.e., by a factor of log n) than optimization.

Consider, for example, the maximum flow problem. Let v_k denote the maximum flow value for problem P_k and x_k denote an arc flow corresponding to v_k. In the problem P_k, the capacity of an arc is twice its capacity in P_{k-1} plus 0 or 1. If we multiply the optimum flow x_{k-1} for P_{k-1} by 2, we obtain a feasible flow for P_k. Moreover, $v_k - 2v_{k-1} \leq m$ because multiplying the flow x_{k-1} by 2 takes care of the doubling of the capacities and the additional 1's can increase the maximum flow value by at most m units (if we add 1 to the capacity of any arc, then we increase the maximum flow from source to sink by at most 1). In general, it is easier to reoptimize such a maximum flow problem. For example, the classical labeling algorithm as discussed in Section 4.1 would perform the reoptimization in at most m augmentations, taking $O(m^2)$ time. Therefore, the scaling version of the labeling algorithm runs in $O(m^2 \log U)$ time, whereas the non-scaling version runs in O(nmU) time. The former time bound is polynomial and the latter bound is only pseudopolynomial. Thus this simple scaling algorithm improves the running time dramatically.

2. BASIC PROPERTIES OF NETWORK FLOWS

As a prelude to the rest of this chapter, in this section we describe several basic properties of network flows. We begin by showing how network flow problems can be modeled in either of two equivalent ways: as flows on arcs as in our formulation in Section 1.1 or as flows on paths and cycles. Then we partially characterize optimal solutions to network flow problems and demonstrate that these problems always have certain special types of optimal solutions (so-called cycle free and spanning tree solutions). Consequently, in designing algorithms, we need only consider these special types of solutions. We next establish several important connections between network flows and linear and integer programming. Finally, we discuss a few useful transformations of network flow problems.

2.1 Flow Decomposition Properties and Optimality Conditions

It is natural to view network flow problems in either of two ways: as flows on arcs or as flows on paths and cycles. In the context of developing underlying theory, models, or algorithms, each view has its own advantages. Therefore, as the first step in our discussion, we will find it worthwhile to develop several connections between these alternate formulations.

In the arc formulation (1.1), the basic decision variables are flows x_{ij} on arcs (i, j). The path and cycle formulation starts with an enumeration of the paths P and cycles Q of the network. Its decision variables are h(p), the flow on path p, and f(q), the flow on cycle q, which are defined for every directed path p in P and every directed cycle q in Q.

Notice that every set of path and cycle flows uniquely determines arc flows in a natural way: the flow x_{ij} on arc (i, j) equals the sum of the flows h(p) and f(q) for all paths p and cycles q that contain this arc. We formalize this observation by defining some new notation: $\delta_{ij}(p)$ equals 1 if arc (i, j) is contained in path p and 0 otherwise; similarly, $\delta_{ij}(q)$ equals 1 if arc (i, j) is contained in cycle q and is 0 otherwise. Then

$$x_{ij} = \sum_{p \in P} \delta_{ij}(p)\, h(p) + \sum_{q \in Q} \delta_{ij}(q)\, f(q).$$

If the flow vector x is expressed in this way, we say that the flow is represented as path flows and cycle flows and that the path flow vector h and cycle flow vector f is a path and cycle flow representation of the flow.

Can we reverse this process? That is, can we decompose any arc flow into (i.e., represent it as) path and cycle flows? The following result provides an affirmative answer to this question.

Theorem 2.1: Flow Decomposition Property (Directed Case). Every directed path and cycle flow has a unique representation as nonnegative arc flows. Conversely, every nonnegative arc flow x can be represented as a directed path and cycle flow (though not necessarily uniquely) with the following two properties:

C2.1. Every path with positive flow connects a supply node of x to a demand node of x.

C2.2. At most n+m paths and cycles have nonzero flow; out of these, at most m cycles have nonzero flow.

Proof. In the light of our previous observations, we need to establish only the converse assertions. We give an algorithmic proof to show that any feasible arc flow x can be decomposed into path and cycle flows. Suppose i_0 is a supply node. Then some arc (i_0, i_1) carries a positive flow. If i_1 is a demand node then we stop; otherwise the mass balance constraint (1.1b) of node i_1 implies that some other arc (i_1, i_2) carries positive flow. We repeat this argument until either we encounter a demand node or we revisit a previously examined node. Note that one of these cases will occur within n steps. In the former case we obtain a directed path p from the supply node i_0 to some demand node i_k consisting solely of arcs with positive flow, and in the latter case we obtain a directed cycle q. If we obtain a directed path, we let $h(p) = \min [b(i_0), -b(i_k), \min \{x_{ij} : (i, j) \in p\}]$, and redefine $b(i_0) = b(i_0) - h(p)$, $b(i_k) = b(i_k) + h(p)$ and $x_{ij} = x_{ij} - h(p)$ for each arc (i, j) in p. If we obtain a cycle q, we let $f(q) = \min \{x_{ij} : (i, j) \in q\}$ and redefine $x_{ij} = x_{ij} - f(q)$ for each arc (i, j) in q.

We repeat this process with the redefined problem until the network contains no supply node (and hence no demand node). Then we select a transhipment node with at least one outgoing arc with positive flow as the starting node, and repeat the procedure, which in this case must find a cycle. We terminate when for the redefined problem x = 0. Clearly, the original flow is the sum of flows on the paths and cycles identified by the procedure. Now observe that each time we identify a path, we reduce the supply/demand of some node or the flow on some arc to zero; and each time we identify a cycle, we reduce the flow on some arc to zero. Consequently, the path and cycle

representation of the given flow x contains at most (n + m) total paths and cycles, of which there are at most m cycles. ■

It is possible to state the decomposition property in a somewhat more general form that permits arc flows x_{ij} to be negative. In this case, even though the underlying network is directed, the paths and cycles can be undirected, and can contain arcs with negative flows. Each undirected path p, which has an orientation from its initial to its final node, has *forward* arcs and *backward arcs* which are defined as arcs along and opposite to the path's orientation. A *path flow* will be defined on p as a flow with value h(p) on each forward arc and -h(p) on each backward arc. We define a *cycle flow* in the same way. In this more general setting, our representation using the notation $\delta_{ij}(p)$ and $\delta_{ij}(q)$ is still valid with the following provision: we now define $\delta_{ij}(p)$ and $\delta_{ij}(q)$ to be -1 if arc (i, j) is a backward arc of the path or cycle.

Theorem 2.2: Flow Decomposition Property (Undirected Case). *Every path and cycle flow has a unique representation as arc flows. Conversely, every arc flow x can be represented as an (undirected) path and cycle flow (though not necessarily uniquely) with the following three properties:*

C2.3. Every path with positive flow connects a source node of x to a sink node of x.

C2.4. For every path and cycle, any arc with positive flow occurs as a forward arc and any arc with negative flow occurs as a backward arc.

C2.5. At most n+m paths and cycles have nonzero flow; out of these, at most m cycles have nonzero flow.

Proof. This proof is similar to that of Theorem 2.1. The major modification is that we extend the path at some node i_{k-1} by adding an arc (i_{k-1}, i_k) with positive flow or an arc (i_k, i_{k-1}) with negative flow. The other steps can be modified accordingly. ■

The flow decomposition property has a number of important consequences. As one example, it enables us to compare any two solutions of a network flow problem in a particularly convenient way and to show how we can *build* one solution from another by a sequence of simple operations.

We need the concept of augmenting cycles with respect to a flow x. A cycle q with flow f(q) > 0 is called an *augmenting cycle* with respect to a flow x if

$$0 \le x_{ij} + \delta_{ij}(q)\, f(q) \le u_{ij}, \text{ for each arc } (i, j) \in q.$$

In other words, the flow remains feasible if some positive amount of flow (namely f(q)) is *augmented* around the cycle q. We define the cost of an augmenting cycle q as $c(q) = \sum_{(i,j) \in A} c_{ij} \delta_{ij}(q)$. The cost of an augmenting cycle represents the change in cost of a feasible solution if we augment along the cycle with one unit of flow. The change in flow cost for augmenting around cycle q with flow f(q) is c(q) f(q).

Suppose that x and y are any two solutions to a network flow problem, i.e., $Nx = b$, $0 \le x \le u$ and $Ny = b$, $0 \le y \le u$. Then the difference vector $z = y - x$ satisfies the homogeneous equations $Nz = Ny - Nx = 0$. Consequently, flow decomposition implies that z can be represented as cycle flows, i.e., we can find at most $r \le m$ cycle flows $f(q_1)$, $f(q_2)$, ... , $f(q_r)$ satisfying the property that for each arc (i, j) of A,

$$z_{ij} = \delta_{ij}(q_1) f(q_1) + \delta_{ij}(q_2) f(q_2) + \ldots + \delta_{ij}(q_r) f(q_r).$$

Since $y = x + z$, for any arc (i, j) we have

$$0 \le y_{ij} = x_{ij} + \delta_{ij}(q_1) f(q_1) + \delta_{ij}(q_2) f(q_2) + \ldots + \delta_{ij}(q_r) f(q_r) \le u_{ij}.$$

Now by condition C2.4 of the flow decomposition property, arc (i, j) is either a forward arc on each cycle $q_1, q_2, \ldots, q_m$ that contains it or a backward arc on each cycle $q_1, q_2, \ldots, q_m$ that contains it. Therefore, each term between x_{ij} and the rightmost inequality in this expression has the same sign; moreover, $0 \le y_{ij} \le u_{ij}$. Consequently, for each cycle q_k, $0 \le x_{ij} + \delta_{ij}(q_k) f(q_k) \le u_{ij}$ for each arc $(i, j) \in q_k$. That is, if we add any of these cycle flows q_k to x, the resulting solution remains feasible on each arc (i, j). Hence, each cycle $q_1, q_2, \ldots, q_r$ is an augmenting cycle with respect to the flow x. Further, note that

$$\sum_{(i,j) \in A} c_{ij} y_{ij} = \sum_{(i,j) \in A} c_{ij} x_{ij} + \sum_{(i,j) \in A} c_{ij} z_{ij}$$

$$= \sum_{(i,j) \in A} c_{ij} x_{ij} + \sum_{(i,j) \in A} c_{ij} \left(\sum_{k=1}^{r} \delta_{ij}(q_k) f(q_k) \right)$$

$$= \sum_{(i,j) \in A} c_{ij} x_{ij} + \sum_{k=1}^{r} c(q_k) f(q_k).$$

We have thus established the following important result.

Theorem 2.3: Augmenting Cycle Property. Let x and y be any two feasible solutions of a network flow problem. Then y equals x plus the flow on at most m augmenting cycles with respect to x. Further, the cost of y equals the cost of x plus the cost of flow on the augmenting cycles. ■

The augmenting cycle property permits us to formulate optimality conditions for characterizing the optimum solution of the minimum cost flow problem. Suppose that x is any feasible solution, that x^* is an optimum solution of the minimum cost flow problem, and that $x \neq x^*$. The augmenting cycle property implies that the difference vector $x^* - x$ can be decomposed into at most m augmenting cycles and the sum of the costs of these cycles equals $cx^* - cx$. If $cx^* < cx$ then one of these cycles must have a negative cost. Further, if every augmenting cycle in the decomposition of $x^* - x$ has a nonnegative cost, then $cx^* - cx \geq 0$. Since x^* is an optimum flow, $cx^* = cx$ and x is also an optimum flow. We have thus obtained the following result.

Theorem 2.4. Optimality Conditions. A feasible flow x is an optimum flow if and only if it admits no negative cost augmenting cycle. ■

2.2 Cycle Free and Spanning Tree Solutions

We start by assuming that x is a feasible solution to the network flow problem

$$\text{minimize } \{ cx : Nx = b \text{ and } l \leq x \leq u \}$$

and that $l = 0$. Much of the underlying theory of network flows stems from a simple observation concerning the example in Figure 2.1. In the example, arc flows and costs are given besides each arc.

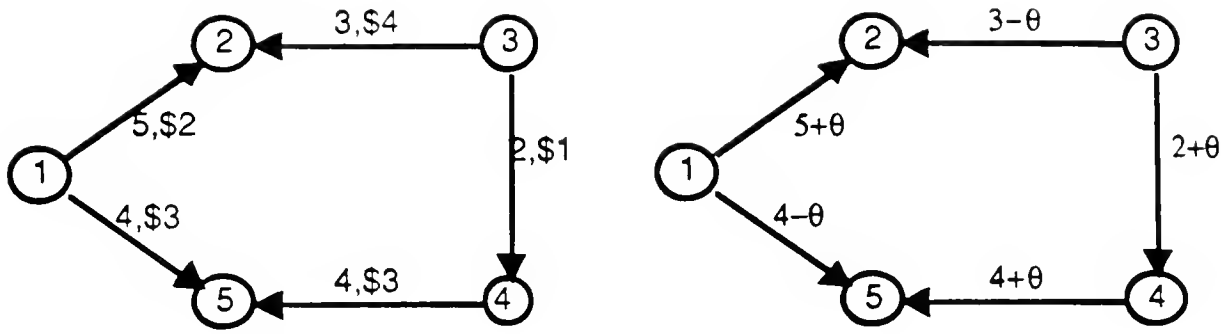


Figure 2.1. Improving flow around a cycle.

Let us assume for the time being that all arcs are uncapacitated. The network in this figure contains flow around an undirected cycle. Note that adding a given amount of flow θ to all the arcs pointing in a clockwise direction and subtracting this flow from all arcs pointing in the counterclockwise direction preserves the mass balance at each node. Also, note that the per unit incremental cost for this flow change is the sum of the cost of the clockwise arcs minus the sum of the cost of counterclockwise arcs, i.e.,

$$\text{Per unit change in cost} = \Delta = \$2 + \$1 + \$3 - \$4 - \$3 = \$ -1.$$

Let us refer to this incremental cost Δ as the *cycle cost* and say that the cycle is a *negative, positive* or *zero cost* cycle depending upon the sign of Δ. Consequently, to minimize cost in our example, we set θ as large as possible while preserving nonnegativity of all arc flows, i.e., $3 - \theta \geq 0$ and $4 - \theta \geq 0$, or $\theta \leq 3$; that is, we set $\theta = 3$. Note that in the new solution (at $\theta = 3$), we no longer have positive flow on all arcs in the cycle.

Similarly, if the cycle cost were positive (i.e., we were to change c_{12} from 2 to 4), then we would decrease θ as much as possible (i.e., $5 + \theta \geq 0$, $2 + \theta \geq 0$, and $4 + \theta \geq 0$, or $\theta \geq -2$) and again find a lower cost solution with the flow on at least one arc in the cycle at value zero. We can restate this observation in another way: to preserve nonnegativity of all flows, we must select θ in the interval $-2 \leq \theta \leq 3$. Since the objective function depends linearly on θ, we optimize it by selecting $\theta = 3$ or $\theta = -2$ at which point one arc in the cycle has a flow value of zero.

We can extend this observation in several ways:

(i) If the per unit cycle cost $\Delta = 0$, we are indifferent to all solutions in the interval $-2 \le \theta \le 3$ and therefore can again choose a solution as good as the original one but with the flow of at least arc in the cycle at value zero.

(ii) If we impose upper bounds on the flow, e.g., such as 6 units on all arcs, then the range of flows that preserves feasibility (i.e., mass balances, lower and upper bounds on flows) is again an interval, in this case $-2 \le \theta \le 1$, and we can find a solution as good as the original one by choosing $\theta = -2$ or $\theta = 1$. At these values of θ, the solution is cycle free, that is, for some arc on the cycle, either the flow is zero (the lower bound) or is at its upper bound ($x_{12} = 6$ at $\theta = 1$).

Some additional notation will be helpful in encapsulating and summarizing our observations up to this point. Let us say that an arc (i, j) is a *free arc* with respect to a given feasible flow x if x_{ij} lies strictly between the lower and upper bounds imposed upon it. We will also say that arc (i, j) is *restricted* if its flow x_{ij} equals either its lower or upper bound. In this terminology, a solution x has the "cycle free property" if the network contains no cycle made up entirely of free arcs.

In general, our prior observations apply to *any* cycle in a network. Therefore, given any initial flow we can apply our previous argument repeatedly, one cycle at a time, and establish the following fundamental result:

Theorem 2.5: Cycle Free Property. If the objective function value of the network optimization problem

$$\textit{minimize } \{ cx : Nx = b, l \le x \le u \}$$

is bounded from below on the feasible region and the problem has a feasible solution, then at least one cycle free solution solves the problem. ■

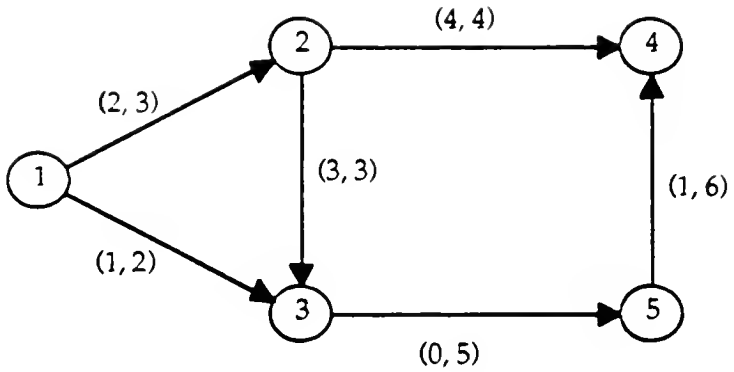
Note that the lower bound assumption imposed upon the objective value is necessary to rule out situations in which the flow change variable θ in our prior argument can be made arbitrarily large in a negative cost cycle, or arbitrarily small (negative) in a positive cost cycle; for example, this condition rules out any negative cost directed cycle with no upper bounds on its arc flows.

It is useful to interpret the cycle free property in another way. Suppose that the network is connected (i.e., there is an undirected path connecting every two pairs of nodes). Then, either a given cycle free solution x contains a free arc that is incident to each node in the network, or we can add to the free arcs some restricted arcs so that the resulting set S of arcs has the following three properties:

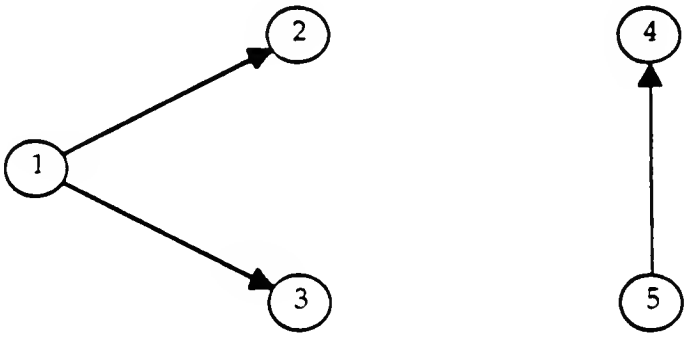
(i) S contains all the free arcs in the current solution,
(ii) S contains no undirected cycles, and
(iii) No superset of S satisfies properties (i) and (ii).

We will refer to any set S of arcs satisfying (i) through (iii) as a *spanning tree* of the network and any feasible solution x for the network together with a spanning tree S that contains all free arcs as a *spanning tree solution.* (At times we will also refer to a given cycle free solution x as a spanning tree solution, with the understanding that restricted arcs may be needed to form the spanning tree S.)

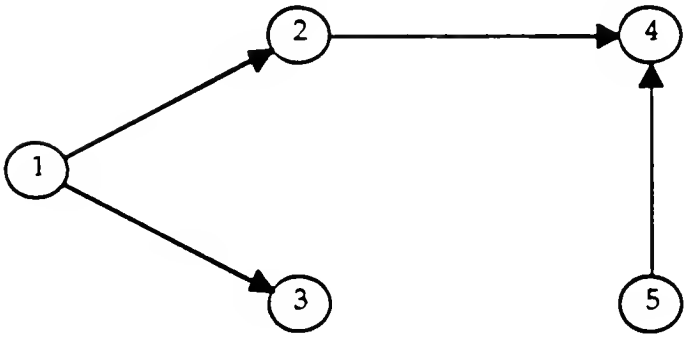
Figure 2.2. illustrates a spanning tree corresponding to a cycle free solution. Note that it may be possible (and often is) to complete the set of free arcs into a spanning tree in several ways (e.g., replace arc (2, 4) with arc (3, 5) in Figure 2.2(c)); therefore, a given cycle free solution can correspond to several spanning trees S. We will say that a spanning tree solution x is *nondegenerate* if the set of free arcs forms a spanning tree. In this case, the spanning tree S corresponding to the flow x is unique. If the free arcs do not span (i.e., are not incident to) all the nodes, then any spanning tree corresponding to this solution will contain at least one arc whose flow equals the arc's lower or upper bound of the arc. In this case, we will say that the spanning tree is *degenerate.*



(a) An example network with arc flows and capacities represented as (x_{ij}, u_{ij}).



(b) A cycle free solution.



(c) A spanning tree solution.

Figure 2.2. Converting a cycle free solution to a spanning tree solution.

When restated in the terminology of spanning trees, the cycle free property becomes another fundamental result in network flow theory.

Theorem 2.6: Spanning Tree Property. If the objective function value of the network optimization problem

$$\text{minimize } \{cx: \ Nx = b, l \le x \le u\}$$

is bounded from below on the feasible region and the problem has a feasible solution then at least one spanning tree solution solves the problem. ■

We might note that the spanning tree property is valid for concave cost versions of the flow problem as well, i.e., those versions where the objective function is a concave function of the flow vector x. This extended version of the spanning tree property is valid because if the incremental cost of a cycle is negative at some point, then the incremental cost remains negative (by concavity) as we augment positive amount of flow around the cycle. Hence, we can increase flow in a negative cost cycle until at least one arc reaches its lower or upper bound.

2.3 Networks, Linear and Integer Programming

The cycle free property and spanning tree property have many other important consequences. In particular, these two properties imply that network flow theory lies at the cusp between two large and important subfields of optimization--linear and integer programming. This positioning may, to a large extent, account for the emergence of network flow theory as a cornerstone of mathematical programming.

Triangularity Property

Before establishing our first results relating network flows to linear and integer programming, we first make a few observations. Note that any spanning tree S has at least one (actually at least two) leaf nodes, that is, a node that is incident to only one arc in the spanning tree. Consequently, if we rearrange the rows and columns of the node-arc incidence matrix of S so that the leaf node is row 1 and its incident arc is column 1, then row 1 has only a single nonzero entry, a +1 or a -1, which lies on the diagonal of the node-arc incidence matrix. If we now remove this leaf node and its incident arc from S, the resulting network is a spanning tree on the remaining nodes. Consequently, by rearranging all but row and column 1 of the node-arc incidence matrix for the spanning tree, we can now assume that row 2 has +1 or -1 element on the

diagonal and zeros to the right of the diagonal. Continuing in this way permits us to rearrange the node-arc incidence matrix of the spanning tree so that its first n-1 rows is lower triangular. Figure 2.3 shows the resulting lower triangular form (actually, one of several possibilities) for the spanning tree in Figure 2.2(c).

$$L = \begin{array}{c} \\ \text{nodes} \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{array} \begin{array}{c} \text{arcs} \\ (5,4) \quad (2,4) \quad (1,3) \quad (1,2) \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

Figure 2.3 The lower triangular form corresponding to a basis.

The node-arc incidence matrix of any spanning tree contains one more row than it has columns (it has n rows and n-1 columns). Therefore, after we have rearranged the matrix so that the first n-1 rows are in triangular form, the node-arc incidence matrix contains one additional row. We will, however, adopt the convention of still referring to it as lower triangular. (As we have noted previously, since each column of any node-arc incidence matrix contains exactly one +1 and one -1, the rows always sum to zero--equivalently, the last row equals -1 times the sum of the other rows and, therefore, is redundant.)

Theorem 2.7: Triangularity Property. The rows and columns of the node-arc incident matrix of any spanning tree can be rearranged to be lower triangular. ■

Integrality of Optimal Solutions

The triangularity property has several important consequences. First, let us evaluate the flows on arcs in a spanning tree solution x. By rearranging rows and columns, we partition the node-arc incident matrix **N** of the network as **N** = [**L**,**M**], where **L** is a lower triangular matrix corresponding to a spanning tree. Suppose that $x = (x^1, x^2)$ is partitioned compatibly. Then

$$\mathbf{N}x = \mathbf{L}x^1 + \mathbf{M}x^2 = b$$
$$\text{or } \mathbf{L}x^1 = b - \mathbf{M}x^2. \qquad (2.1)$$

Now further suppose that the supply/demand vector b and lower and upper bound vectors *l* and u have all integer components. Then since every component of x^2 equals an arc lower or upper bound and M has integer components (each equal to 0, +1, or -1), the right hand side $b - Mx^2$ is an integer vector. But this observation implies that the components of x^1 are integral as well: since the first diagonal element of U equals +1 or -1, the first equation in (2.1) implies that x_1^1 is integral; now if we move x_1^1 to the right of the equality in (2.1), the right hand side remains integral and we can solve for x_2^1 from the second equation; continuing this forward substitution by successively solving for one variable at a time shows that x^1 is integral.

This argument shows that for problems with integral data, every spanning tree solution is integral. Since the spanning tree property ensures that network flow problems always have spanning tree solutions, we have established the following fundamental result.

Theorem 2.8. Integrality Property. *If the objective value of the network optimization problem*

$$\textit{minimize } \{\, cx: Nx = b, l \le x \le u \,\}$$

is bounded from below on the feasible region, the problem has a feasible solution, and the vectors b, l, and u are integer, then the problem has at least one integer optimum solution. ■

Our observation at the end of Section 2.2 shows that this integrality property is also valid in the more general situation in which the objective function is concave.

Relationship to Linear Programming

The network flow problem with the objective function cx is a linear program which, as the last result shows, always has an integer solution. Network flow problems are distinguished as the most important large class of problems with this property.

Linear programs, or generalizations with concave cost objective functions, also satisfy another well-known property: they always have, in the parlance of convex analysis, *extreme point solutions*; that is, solutions x with the property that x cannot be expressed as a weighted combination of two other feasible solutions y and z, i.e., as $x = \alpha y + (1-\alpha)z$ for some weight $0 < \alpha < 1$. Since, as we have seen, network flow problems always have cycle free solutions, we might expect to discover that extreme point

solutions and cycle free solutions are closely related, and indeed they are as shown by the next result.

Theorem 2.9. Extreme Point Property. For network flow problems, every cycle free solution is an extreme point and, conversely, every extreme point is a cycle free solution. Consequently, if the objective value of the network optimization problem

$$\text{minimize } \{ cx: \; Nx = b, l \le x \le u \}$$

is bounded from below on the feasible region and the problem has a feasible solution, then the problem has an extreme point solution.

Proof. With the background developed already, this result is easy to establish. First, if x is not a cycle free solution, then it cannot be an extreme point, since by perturbing the flow by a small amount θ and by a small amount $-\theta$ around a cycle with free arcs, as in our discussion of Figure 2.1, we define two feasible solutions y and z with the property that $x = (1/2)y + (1/2)z$. Conversely, suppose that x is not an extreme point and is represented as $x = \alpha y + (1-\alpha)z$ with $0 < \alpha < 1$. Let x^1, y^1 and z^1 be the components of these vectors for which y and z differ, i.e., $l_{ij} \le y_{ij} < x_{ij} < z_{ij} \le u_{ij}$ or $l_{ij} \le z_{ij} < x_{ij} < y_{ij} \le u_{ij}$, and let N_1 denote the submatrix of N corresponding to these arcs (i, j). Then $N_1(z^1 - y^1) = 0$, which implies, by flow decomposition, that the network contains an undirected cycle with y_{ij} not equal to z_{ij} for any arc on the cycle. But by definition of the components x^1, y^1, and z^1, this cycle contains only free arcs in the solution x. Therefore, if x is not an extreme point solution, then it is not a cycle free solution. ■

In linear programming, extreme points are usually represented algebraically as *basic solutions;* for these special solutions, the columns B of the constraint matrix of a linear program corresponding to variables strictly between their lower and upper bounds are linearly independent. We can extend B to a basis of the constraint matrix by adding a maximal number of columns. Just as cycle free solutions for network flow problems correspond to extreme points, spanning tree solutions correspond to basic solutions.

Theorem 2.10: Basis Property. Every spanning tree solution to a network flow problem is a basic solution and, conversely, every basic solution is a spanning tree solution. ■

Let us now make one final connection between networks and linear and integer programming--namely, between basis and the integrality property. Consider a linear program of the form Ax = b and suppose that N = [B,M] for some basis B and that $x = (x^1,x^2)$ is a compatible partitioning of x. Also suppose that we eliminate the redundant row so that B is a nonsingular matrix. Then

$$Bx^1 = b - Mx^2, \text{ or } x^1 = B^{-1}(b - Mx^2).$$

Also, by Cramer's rule from linear algebra, it is possible to find each component of x^1 as sums and multiples of components of $b' = b - Mx^2$ and B, divided by det(B), the determinant of B. Therefore, if the determinant of B equals +1 or -1, then x^1 is an integer vector whenever x^2, b, and M are composed of all integers. In particular, if the partitioning of A corresponds to a basic feasible solution x and the problem data A, b, *l* and u are all integers, then x^2 and consequently x^1 is an integer. Let us call a matrix A *unimodular* if all of its bases have determinants either +1 or -1, and call it *totally unimodular* if all of its square submatrices have determinant equal to either 0,+1, or -1.

How are these notions related to network flows and the integrality property? Since bases of N correspond to spanning trees, the triangularity property shows that the determinant of any basis (excluding the redundant row now), equals the product of the diagonal elements in the triangular representation of the basis, and therefore equals +1 or -1. Consequently, a node-arc incident matrix is unimodular. Even more, it is totally unimodular. For let S be any square submatrix of N. If S is singular, it has determinant 0. Otherwise, it must correspond to a cycle free solution, which is a spanning tree on each of its connected components. But then, it is easy to see that the determinant of S is the product of the determinants of the spanning trees and, therefore, it must be equal to +1 or -1. (An induction argument, using an expansion of determinants by minors, provides an alternate proof of this totally unimodular property.)

Theorem 2.11: Total Unimodularity Property. The constraint matrix of a minimum cost network flow problem is totally unimodular. ■

2.4 Network Transformations

Frequently, analysts use network transformations to simplify a network problem, to show equivalences of different network problems, or to put a network problem into a standard form required by a computer code. In this subsection, we describe some of these important transformations.

T1. (Removing Nonzero Lower Bounds). If an arc (i, j) has a positive lower bound l_{ij}, then we can replace x_{ij} by $x'_{ij} + l_{ij}$ in the problem formulation. As measured by the new variable x'_{ij}, the flow on arc (i, j) will have a lower bound of 0. This transformation has a

simple network interpretation: we begin by sending l_{ij} units of flow on the arc and then measure incremental flow above l_{ij}.

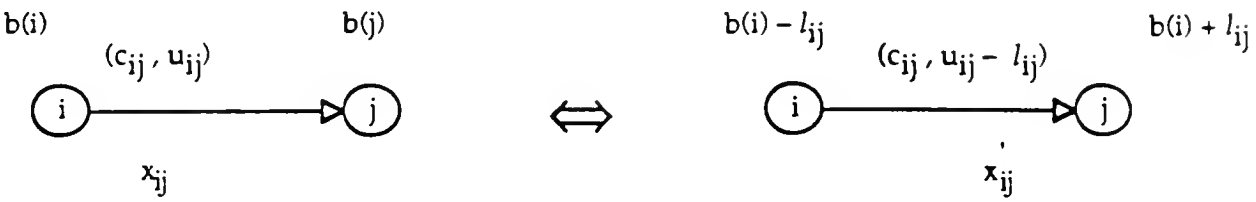


Figure 2.4. Transforming a positive lower bound to zero.

T2. (Removing Capacities). If an arc (i, j) has a positive capacity u_{ij}, then we can remove the capacity, making the arc uncapacitated, using the following ideas. The capacity constraint (i, j) can be written as $x_{ij} + s_{ij} = u_{ij}$, if we introduce a slack variable $s_{ij} \geq 0$. Multiplying both sides by -1, we obtain

$$-x_{ij} - s_{ij} = -u_{ij}. \qquad (2.2)$$

This transformation is tantamount to turning the slack variable into an additional node k with equation (2.2) as the mass balance constraint for that node. Observe that the variable x_{ij} now appears in three mass balance constraints and s_{ij} in only one. By subtracting (2.2) from the mass balance constraint of node j, we assure that each of x_{ij} and s_{ij} appear in exactly two constraints--in one with the positive sign and in the other with the negative sign. These algebraic manipulations correspond to the following network transformation.

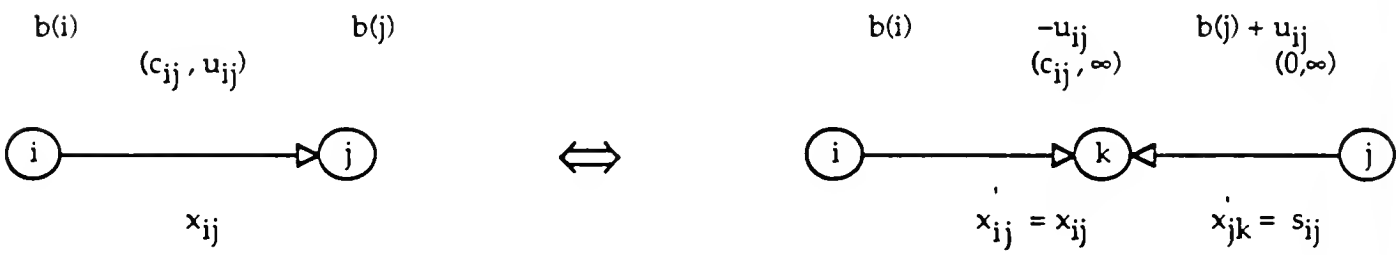


Figure 2.5. Removing arc capacities.

In the network context, this transformation implies the following. If x_{ij} is a flow on arc (i, j) in the original network, the corresponding flow in the transformed network is $x'_{ik} = x_{ij}$ and $x'_{jk} = u_{ij} - x_{ij}$; both the flows x and x' have the same cost. Likewise, a flow x'_{ik}, x'_{jk} in the transformed network yields a flow of $x_{ij} = x'_{jk}$ of the same cost in the

original network. Further, since $x'_{ik} + x'_{jk} = u_{ij}$ and x'_{ik} and x'_{jk} are both nonnegative, $x_{ij} = x'_{ik} \leq u_{ij}$. Consequently, this transformation is valid.

T3. (Arc Reversal). Let u_{ij} represent the capacity of the arc (i, j) or an upper bound on the arc flow if it is uncapacitated. This transformation is a change in variable: replace x_{ij} by $u_{ij} - x'_{ji}$ in the problem formulation. Doing so replaces arc (i, j) with its associated cost c_{ij} by the arc (j, i) with a cost $-c_{ij}$. Therefore, this transformation permits us to remove arcs with negative costs. This transformation has the following network interpretation: send u_{ij} units of flow on the arc and then replace arc (i, j) by arc (j, i) with cost $-c_{ij}$. The new flow x'_{ji} measures the amount of flow we "remove" from the "full capacity" flow of u_{ij}.

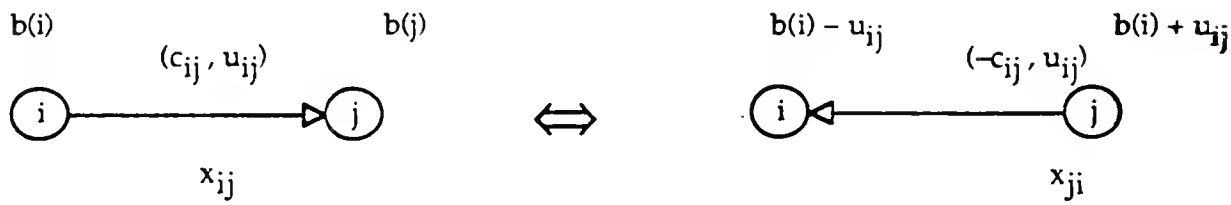


Figure 2.6. An example of arc reversal.

T4. (Node Splitting). This transformation splits each node i into two nodes i and i' and replaces each original arc (k, i) by an arc (k, i') of the same cost and capacity, and each arc (i, j) by an arc (i', j) of the same cost and capacity. We also add arcs (i, i') of cost zero for each i. Figure 2.7 illustrates the resulting network when we carry out the node splitting transformation for all the nodes of a network.

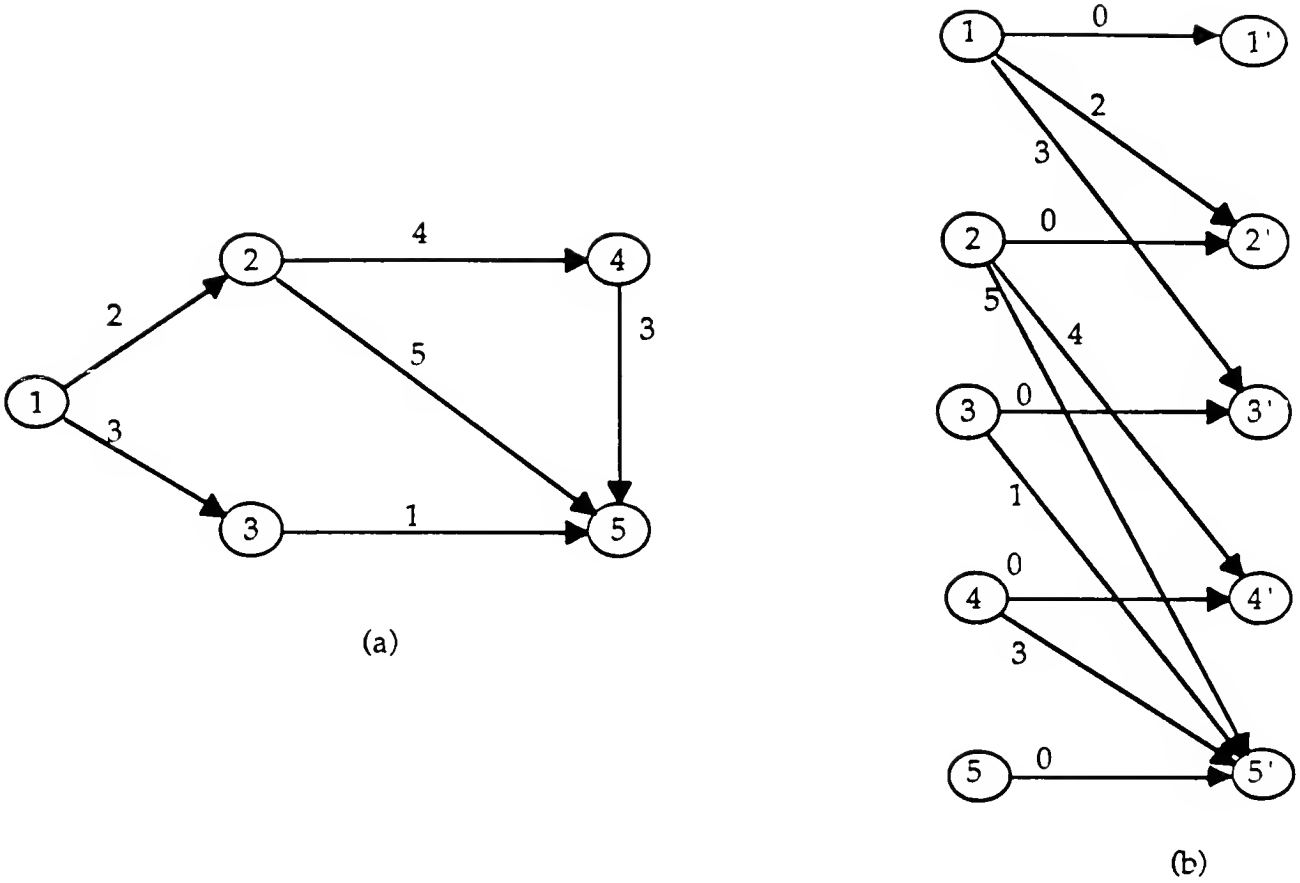


Figure 2.7. (a) The original network. (b) The transformed network.

We shall see the usefulness of this transformation in Section 5.11 when we use it to reduce a shortest path problem with arbitrary arc lengths to an assignment problem. This transformation is also used in practice for representing node activities and node data in the standard "arc flow" form of the network flow problem: we simply associate the cost or capacity for the throughput of node i with the new throughput arc (i, i').

3. SHORTEST PATHS

Shortest path problems are the most fundamental and also the most commonly encountered problems in the study of transportation and communication networks. The shortest path problem arises when trying to determine the shortest, cheapest, or most reliable path between one or many pairs of nodes in a network. More importantly, algorithms for a wide variety of combinatorial optimization problems such as vehicle routing and network design often call for the solution of a large number of shortest path problems as subroutines. Consequently, designing and testing efficient algorithms for the shortest path problem has been a major area of research in network optimization.

Researchers have studied several different (directed) shortest path models. The major types of shortest path problems, in increasing order of solution difficulty, are (i) finding shortest paths from one node to all other nodes when arc lengths are nonnegative; (ii) finding shortest paths from one node to all other nodes for networks with arbitrary arc lengths; (iii) finding shortest paths from every node to every other node; and (iv) finding various types of constrained shortest paths between nodes (e.g., shortest paths with turn penalties, shortest paths visiting specified nodes, the k-th shortest path).

In this section, we discuss problem types (i), (ii) and (iii). The algorithmic approaches for solving problem types (i) and (ii) can be classified into two groups--*label setting* and *label correcting*. The label setting methods are applicable to networks with nonnegative arc lengths, whereas label correcting methods apply to networks with negative arc lengths as well. Each approach assigns tentative distance labels (shortest path distances) to nodes at each step. Label setting methods designate one or more labels as permanent (optimum) at each iteration. Label correcting methods consider all labels as temporary until the final step when they all become permanent. We will show that label setting methods have the most attractive worst-case performance; nevertheless, practical experience has shown the label correcting methods to be modestly more efficient.

Dijkstra's algorithm is the most popular label setting method. In this section, we first discuss a simple implementation of this algorithm that achieves a time bound of $O(n^2)$. We then describe two more sophisticated implementations that achieve improved running times in practice and in theory. Next, we consider a generic version of the label correcting method, outlining one special implementation of this general approach that runs in polynomial time and another implementation that performs very

well in practice. Finally, we discuss a method to solve the all pairs shortest path problem.

3.1 Dijkstra's Algorithm

We consider a network G= (N,A) with an arc length c_{ij} associated with each arc (i, j) $\in$ A. Let A(i) represent the set of arcs emanating from node i $\in$ N, and let C = max { c_{ij} : (i, j) $\in$ A }. In this section, we assume that arc lengths are integer numbers, and in this section as well as in Sections 3.2 and 3.3, we further assume that arc lengths are nonnegative. We suppose that node s is a specially designated node, and assume without any loss of generality that the network G contains a directed path from s to every other node. We can ensure this condition by adding an artificial arc (s, j), with a suitably large arc length, for each node j. We invoke this connectivity assumption throughout this section.

Dijkstra's algorithm finds shortest paths from the source node s to all other nodes. The basic idea of the algorithm is to fan out from node s and label nodes in order of their distances from s. Each node i has a label, denoted by d(i): the label is *permanent* once we know that it represents the shortest distance from s to i; otherwise it is *temporary*. Initially, we give node s a permanent label of zero, and each other node j a temporary label equal to c_{sj} if (s, j) $\in$ A, and ∞ otherwise. At each iteration, the label of a node i is its shortest distance from the source node along a path whose internal nodes are all permanently labeled. The algorithm selects a node i with the minimum temporary label, makes it permanent, and scans arcs in A(i) to update the distance labels of adjacent nodes. The algorithm terminates when it has designated all nodes as permanently labeled. The correctness of the algorithm relies on the key observation (which we prove later) that it is always possible to designate the node with the minimum temporary label as permanent. The following algorithmic representation is a basic implementation of Dijkstra's algorithm.

```
algorithm DIJKSTRA;
begin
        P := {s} ; T := N – {s};
        d(s) := 0 and pred(s) := 0;
        d(j) := c_sj and pred(j) := s if (s,j) ∈ A , and d(j) := ∞ otherwise;
        while P ≠ N do
        begin
        (node selection) let i ∈ T be a node for which d(i) = min {d(j) : j ∈ T};
                                P := P ∪ {i}; T: = T – {i};
        (distance update) for each (i,j) ∈ A(i) do
                                if d(j) > d(i) + c_ij then d(j) := d(i) + c_ij and pred(j) := i;
        end;
end;
```

The algorithm associates a predecessor index, denoted by *pred(i)*, with each node $i \in N$. The algorithm updates these indices to ensure that *pred(i)* is the last node prior to i on the (tentative) shortest path from node s to node i. At termination, these indices allow us to trace back along a shortest path from each node to the source.

To establish the validity of Dijkstra's algorithm, we use an inductive argument. At each point in the algorithm, the nodes are partitioned into two sets, P and T. Assume that the label of each node in P is the length of a shortest path from the source, whereas the label of each node j in T is the length of a shortest path subject to the restriction that each node in the path (except j) belongs to P. Then it is possible to transfer the node i in T with the smallest label d(i) to P for the following reason: any path P from the source to node i must contain a first node k that is in T. However, node k must be at least as far away from the source as node i since its label is at least that of node i; furthermore, the segment of the path P between node k and node i has a nonnegative length because arc lengths are nonnegative. This observation shows that the length of path P is at least d(i) and hence it is valid to permanently label node i. After the algorithm has permanently labeled node i, the temporary labels of some nodes in $T - \{i\}$ might decrease, because node i could become an internal node in the tentative shortest paths to these nodes. We must thus scan all of the arcs (i, j) in A(i); if $d(j) > d(i) + c_{ij}$, then setting $d(j) = d(i) + c_{ij}$ updates the labels of nodes in $T - \{i\}$.

The computational time for this algorithm can be split into the time required by its two basic operations--selecting nodes and updating distances. In an iteration, the algorithm requires O(n) time to identify the node i with minimum temporary label and

takes $O(|A(i)|)$ time to update the distance labels of adjacent nodes. Thus, overall, the algorithm requires $O(n^2)$ time for selecting nodes and $O(\sum_{i \in N} |A(i)|) = O(m)$ time for updating distances. *This implementation of Dijkstra's algorithm thus runs in $O(n^2)$ time.*

Dijkstra's algorithm has been a subject of much research. Researchers have attempted to reduce the node selection time without substantially increasing the time for updating distances. Consequently, they have, using clever data structures, suggested several implementations of the algorithm. These implementations have either dramatically reduced the running time of the algorithm in practice or improved its worst case complexity. In the following discussion, we describe Dial's algorithm, which is currently comparable to the best label setting algorithm in practice. Subsequently we describe an implementation using R-heaps, which is nearly the best known implementation of Dijkstra's algorithm from the perspective of worst-case analysis. (A more complex version of R-heaps gives the best worst-case performance for most all choices of the parameters n, m, and C.)

3.2 Dial's Implementation

The bottleneck operation in Dijkstra's algorithm is node selection. To improve the algorithm's performance, we must ask the following question. Instead of scanning all temporarily labeled nodes at each iteration to find the one with the minimum distance label, can we reduce the computation time by maintaining distances in a sorted fashion? Dial's algorithm tries to accomplish this objective, and reduces the algorithm's computation time in practice, using the following fact:

FACT 3.1. *The distance labels that Dijkstra's algorithm designates as permanent are nondecreasing.*

This fact follows from the observation that the algorithm permanently labels a node i with smallest temporary label d(i), and while scanning arcs in A(i) during the distance update step, never decreases the distance label of any permanently labeled node since arc lengths are nonnegative. FACT 3.1 suggests the following scheme for node selection. We maintain nC+1 *buckets* numbered 0, 1, 2, ... , nC. Bucket k stores each node whose temporary distance label is k. Recall that C represents the largest arc length in the network and, hence, nC is an upper bound on the distance labels of all the nodes. In the node selection step, we scan the buckets in increasing order until we identify the first nonempty bucket. The distance label of each node in this bucket is minimum. One by

one, we delete these nodes from the bucket, making them permanent and scanning their arc lists to update distance labels of adjacent nodes. We then resume the scanning of higher numbered buckets in increasing order to select the next nonempty bucket.

By storing the content of these buckets carefully, it is possible to add, delete, and select the next element of any bucket very efficiently; in fact, in O(1) time, i.e., a time bounded by some constant. One implemention uses a data structure known as a *doubly linked list*. In this data structure, we order the content of each bucket arbitrarily, storing two pointers for each entry: one pointer to its immediate predecessor and one to its immediate successor. Doing so permits us, by rearranging the pointers, to select easily the topmost node from the list, add a bottommost node, or delete a node. Now, as we relabel nodes and decrease any node's temporary distance label, we move it from a higher index bucket to a lower index bucket; this transfer requires O(1) time. Consequently, this algorithm runs in O(m + nC) time and uses nC+1 buckets. The following fact allows us to reduce the number of buckets to C+1.

FACT 3.2. If d(i) is the distance label that the algorithm designates as permanent at the beginning of an iteration, then at the end of that iteration d(j) ≤ d(i) + C for each finitely labeled node j in T.

This fact follows by noting that (i) $d(k) \le d(i)$ for each $k \in P$ (by FACT 3.1), and (ii) for each finitely labeled node j in T, $d(j) = d(k) + c_{kj}$ for some $k \in P$ (by the property of distance updates). Hence, $d(j) \le d(i) + c_{kj} \le d(i) + C$. In other words, all finite temporary labels are bracketed from below by d(i) and from above by d(i) + C. Consequently, C+1 buckets suffice to store nodes with finite temporary distance labels. We need not store the nodes with infinite temporary distance labels in any of the buckets—we can add them to a bucket when they first receive a finite distance label.

Dial's algorithm uses C+1 buckets numbered 0, 1, 2, ... , C which can be viewed as arranged in a circle as in Figure 3.1. This implementation stores a temporarily labeled node j with distance label d(j) in the bucket d(j) mod (C+1). Consequently, during the entire execution of the algorithm, bucket k stores temporary labeled nodes with distance labels k, k+(C+1), k+2(C+1), and so forth; however, because of FACT 3.2, at any point in time this bucket will hold only nodes with the same distance labels. This storage scheme also implies that if bucket k contains a node with minimum distance label, then buckets k+1, k+2, ... , C, 0, 1, 2, ... , k-1, store nodes in increasing values of the distance labels.

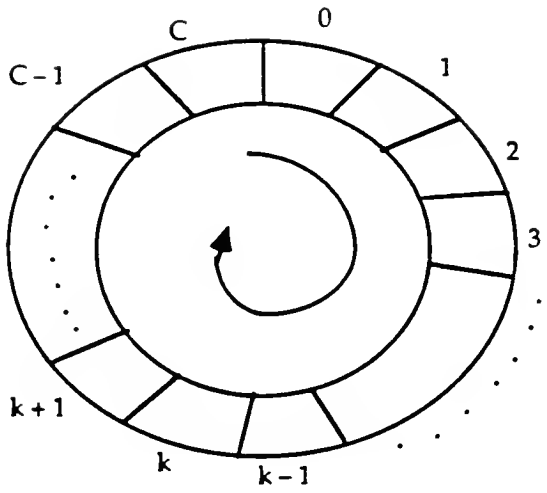


Figure 3.1. Bucket arrangement in Dial's algorithm

Dial's algorithm examines the buckets sequentially, in a wrap around fashion, to identify the first nonempty bucket. In the next iteration, it reexamines the buckets starting at the place where it left off earlier. A potential disadvantage of this scheme, as compared to the original algorithm, is that C may be very large, necessitating large storage and increased computational time. In addition, the algorithm may wrap around as many as n-1 times, resulting in a large computation time. The algorithm, however, typically does not encounter these difficulties in practice. For most applications, C is not very large, and the number of passes through all of the buckets is much less than n. Dial's algorithm, however, is not attractive theoretically. The algorithm runs in $O(m + nC)$ time which is not even polynomial time. Rather, it is pseudopolynomial time. For example, if $C = n^4$, then the algorithm runs in $O(n^5)$ time, and if $C = 2^n$ the algorithm takes exponential time in the worst case.

The search for the theoretically fastest implementations of Dijkstra's algorithm has led researchers to develop several new data structures for sparse networks. In the next section, we consider an implementation using a data structure called a *redistributive heap* (R-heap) that runs in $O(m + n \log nC)$ time. The discussion of this implementation is of a more advanced nature than the previous sections and the reader can skip it without any loss of continuity.

3.3. R-Heap Implementation

Our first $O(n^2)$ implementation of Dijkstra's algorithm and then Dial's implementation represent two extremes. The first implementation considers all the

temporarily labeled nodes together (in one large bucket, so to speak) and searches for a node with the smallest label. Dial's algorithm separates nodes by storing any two nodes with different labels in different buckets. Could we improve upon these methods by adopting an intermediate approach, perhaps by storing many, but not all, labels in a bucket? For example, instead of storing only nodes with a temporary label of k in the k-th bucket, we could store temporary labels from 100k to 100k+99 in bucket k. The different temporary labels that can be stored in a bucket make up the *range* of the bucket; the cardinality of the range is called its *width*. For the preceding example, the range of bucket k is [100k .. 100k+99] and its width is 100.

Using widths of size k permits us to reduce the number of buckets needed by a factor of k. But in order to find the smallest distance label, we need to search all of the elements in the smallest index nonempty bucket. Indeed, if k is arbitrarily large, we need only one bucket, and the resulting algorithm reduces to Dijkstra's original implementation.

Using a width of 100, say, for each bucket reduces the number of buckets, but still requires us to search through the lowest numbered bucket to find the node with minimum temporary label. If we could devise a variable width scheme, with a width of one for the lowest numbered bucket, we could conceivably retain the advantages of both the wide bucket and narrow bucket approaches. The R-heap algorithm we consider next uses variable length widths and changes the ranges dynamically. In the version of redistributive heaps that we present, the widths of the buckets are 1, 1, 2, 4, 8, 16, ... , so that the number of buckets needed is only O(log nC). Moreover, we dynamically modify the ranges of numbers stored in each bucket and we reallocate nodes with temporary distance labels in a way that stores the minimum distance label in a bucket whose width is 1. In this way, as in the previous algorithm, we avoid the need to search the entire bucket to find the minimum. In fact, the running time of this version of the R-heap algorithm is O(m + n log nC).

We now describe an R-heap in more detail. For a given shortest path problem, the R-heap consists of $1 + \lceil \log nC \rceil$ buckets. The buckets are numbered as 0, 1, 2, ... , $K = \lceil \log nC \rceil$. We represent the range of bucket k by *range(k)* which is a (possibly empty) closed interval of integers. We store a temporary node i in bucket k if $d(i) \in$ range(k). We do not store permanent nodes. The nodes in bucket k are denoted by the set CONTENT(k). The algorithm will change the ranges of the buckets dynamically, and each time it changes the ranges, it redistributes the nodes in the buckets.

Initially, the buckets have the following ranges:

range(0) = [0];
range(1) = [1];
range(2) = [2 .. 3];
range(3) = [4 .. 7];
range(4) = [8 .. 15];
.
.
.
range(K) = $[2^{K-1} .. 2^{K}-1]$.

These ranges will change dynamically; however, the widths of the buckets will not increase beyond their initial widths. Suppose for example that the initial minimum distance label is determined to be in the range [8 .. 15]. We could verify this fact quickly by verifying that buckets 0 through 3 are empty and bucket 4 is nonempty. At this point, we could not identify the minimum distance label without searching all nodes in bucket 4. The following observation is helpful. Since the minimum index nonempty bucket is the bucket whose range is [8 .. 15], we know that no temporary label will ever again be less than 8, and hence buckets 0 to 3 will never be needed again. Rather than leaving these buckets idle, we can redistribute the range of bucket 4 (whose width is 8) to the previous buckets (whose combined width is 8) resulting in the ranges [8], [9], [10 .. 11], and [12 .. 15]. We then set the range of bucket 4 to 0, and we shift (or redistribute) its temporarily labeled nodes into the appropriate buckets (0, 1, 2, and 3). Thus, each of the elements of bucket 4 moves to a lower indexed bucket.

Essentially, we have replaced the node selection step (i.e., finding a node with smallest temporary distance label) by a sequence of redistribution steps in which we shift nodes constantly to lower indexed buckets. Roughly speaking, the redistribution time is O(n log nC) time in total, since each node can be shifted at most $K = 1 + \lceil \log nC \rceil$ times. Eventually, the minimum temporary label is in a bucket with width one, and the algorithm selects it in an additional O(1) time.

Actually, we would carry out these operations a bit differently. Since we will be scanning all of the elements of bucket 4 in the redistribute step, it makes sense to first find the minimum temporary label in the bucket. Suppose for example that the minimum label is 11. Then rather than redistributing the range [8 .. 15], we need only redistribute the subrange [11 .. 15]. In this case the resulting ranges of buckets 0 to 4

would be [11], [12], [13 .. 14], [15], ø. Moreover, at the end of this redistribution, we are guaranteed that the minimum temporary label is stored in bucket 0, whose width is 1.

To reiterate, we do not carry out the actual node selection step until the minimum nonempty bucket has width one. If the minimum nonempty bucket is bucket k, whose width is greater than 1, we redistribute the range of bucket k into buckets 0 to k-1, and then we reassign the content of bucket k to buckets 0 to k-1. The redistribution time is O(n log nC) and the running time of the algorithm is O(m + n log nC).

We now illustrate R-heaps on the shortest path example given in Figure 3.2. In the figure, the number beside each arc indicates its length.

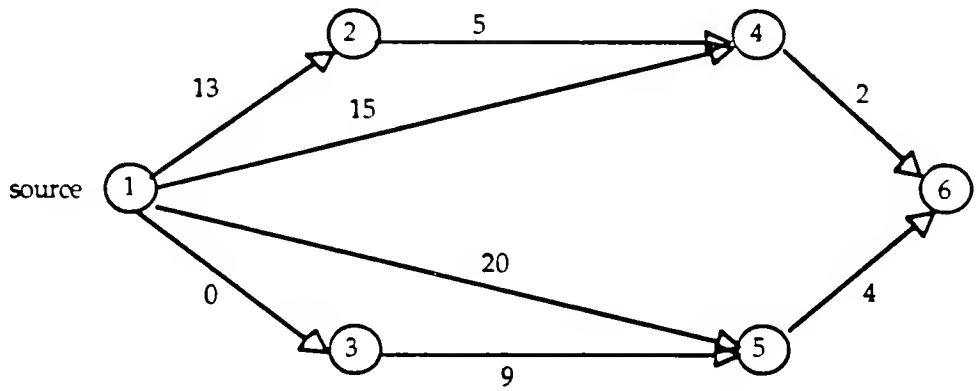


Figure 3.2 The shortest path example.

For this problem, C=20 and K = ⌈log 120⌉ = 7. Figure 3.3 specifies the starting solution of Dijkstra's algorithm and the initial R-heap.

Node i:	1	2	3	4	5	6		
Label d(i):	0	13	0	15	20	nC = 120		
Buckets:	0	1	2	3	4	5	6	7
Ranges:	[0]	[1]	[2 .. 3]	[4 .. 7]	[8 .. 15]	[16 .. 31]	[32 .. 63]	[64 .. 127]
CONTENT:	{3}	ø	ø	ø	{2, 4}	{5}	ø	{6}

Figure 3.3 The initial R-heap.

To select the node with the smallest distance label, we scan the buckets 0, 1, 2, ... , K to find the first nonempty bucket. In our example, bucket 0 is nonempty. Since bucket 0 has width 1, every node in this bucket has the same (minimum) distance label. So, the

algorithm designates node 3 as permanent, deletes node 3 from the R-heap, and scans the arc (3,5) to change the distance label of node 5 from 20 to 9. We check whether the new distance label of node 5 is contained in the range of its present bucket, which is bucket 5. It isn't. Since its distance label has decreased, node 5 should move to a lower index bucket. So we sequentially scan the buckets from right to left, starting at bucket 5, to identify the first bucket whose range contains the number 9, which is bucket 4. Node 5 moves from bucket 5 to bucket 4. Figure 3.4 shows the new R-heap.

Node i:	2	4	5	6				
Label d(i):	13	15	9	120				
Buckets:	0	1	2	3	4	5	6	7
Ranges:	[0]	[1]	[2 .. 3]	[4 .. 7]	[8 .. 15]	[16 .. 31]	[32 .. 63]	[64 .. 127]
CONTENT:	ø	ø	ø	ø	{2, 4, 5}	ø	ø	{6}

Figure 3.4 The R-heap at the end of Iteration 1.

We again look for the node with smallest distance label. Scanning the buckets sequentially, we find that bucket k = 4 is the first nonempty bucket. Since the range of this bucket contains more than one integer, the first node in the bucket need not have the minimum distance label. Since the algorithm will never use the ranges range(0), ... , range(k-1) for storing temporary distance labels, we can redistribute the range of bucket k into the buckets 0, 1, ... , k-1 and reinsert its nodes into the lower indexed buckets. In our example, the range of bucket 4 is [8 .. 15], but the smallest distance label in this bucket is 9. We therefore redistribute the range [9 .. 15] over the lower indexed buckets in the following manner:

range(0) = [9];
range(1) = [10];
range(2) = [11 .. 12];
range(3) = [13 .. 15];
range(4) = ø.

Other ranges do not change. The range of bucket 4 is now empty, and we must reassign the content of bucket 4 to buckets 0 through 3. We do so by successively selecting nodes in bucket 4, sequentially scanning the buckets 3, 2, ... , 0 and inserting the node in the appropriate bucket. The resulting buckets have the following content:

CONTENT(0) = {5},
CONTENT(1) = ø,
CONTENT(2) = ø,
CONTENT(3) = {2. . 4},
CONTENT(4) = ø.

This redistribution necessarily empties bucket 4 , and moves the node with the smallest distance label to bucket 0.

We are now in a position to outline the general algorithm and analyze its complexity. Suppose that j ∈ CONTENT(k) and that d(j) decreases. If the modified d(j) ∉ range(k), then we sequentially scan lower numbered buckets from right to left and add the node to the appropriate bucket. Overall, this operation takes O(m + nK) time. The term m reflects the number of distance updates, and the term nK arises because every time a node moves, it moves to a lower indexed bucket; since there are K+1 buckets, a node can move at most K times. Therefore, O(nK) is a bound on the total node movements.

Next we consider the node selection step. Node selection begins by scanning the buckets from left to right to identify the first nonempty bucket, say bucket k. This operation takes O(K) time per iteration and O(nK) time in total. If k=0 or k=1, then any node in the selected bucket has the minimum distance label. If k ≥ 2, then we redistribute the "useful" range of bucket k into the buckets 0, 1, ... , k-1 and reinsert its content to those buckets. If the range of bucket k is [*l* .. u] and the smallest distance label of a node in the bucket is d_{min}, then the useful range of the bucket is [d_{min} .. u].

The algorithm redistributes the useful range in the following manner: we assign the first integer to bucket 0, the next integer to bucket 1, the next two integers to bucket 2, the next four integers to bucket 3, and so on. Since bucket k has width ≤ 2^k and since the widths of the first buckets can be as large as 1, 1, 2, ... , 2^{k-1} for a total potential width of 2^k, we can redistribute the useful range of bucket k over the buckets 0, 1, ... , k-1 in the manner described. This redistribution of ranges and the subsequent reinsertions of nodes empties bucket k and moves the nodes with the smallest distance labels to bucket 0. Whenever we examine a node in the nonempty bucket k with the smallest index, we move it to a lower indexed bucket; each node can move at most K times, so all the nodes can move a total of at most nK times. Thus, the node selection steps take O(nK) total time. Since K = ⌈log nC⌉ the algorithm runs in O(m + n log nC) time. We now summarize our discussion.

Theorem 3.1. The R-heap implementation of Dijkstra's algorithm solves the shortest path problem in O(m + n log nC) time.

This algorithm requires $1 + \lceil \log nC \rceil$ buckets. FACT 3.2 permits us to reduce the number of buckets to $1 + \lceil \log C \rceil$. This refined implementation of the algorithm runs in O(m + n log C) time. For probelm that satisfy the similarity assumption (see Section 1.2), this bound becomes O(m+ n log n). Using substantially more sophisticated data structures, it is possible to reduce this bound further to $O(m + n \sqrt{\log n})$, which is a linear time algorithm for all but the sparsest classes of shortest path problems.

3.4. Label Correcting Algorithms

Label correcting algorithms, as the name implies, maintain tentative distance labels for nodes and correct the labels at every iteration. Unlike label setting algorithms, these algorithms maintain all distance labels as temporary until the end, when they all become permanent simultaneously. The label correcting algorithms are conceptually more general than the label setting algorithms and are applicable to more general situations, for example, to networks containing negative length arcs. To produce shortest paths, these algorithms typically require that the network does not contain any negative directed cycle, i.e., a directed cycle whose arc lengths sum to a negative value. Most label correcting algorithms have the capability to detect the presence of negative cycles.

Label correcting algorithms can be viewed as a procedure for solving the following recursive equations:

$$d(s) = 0; \tag{3.1}$$

$$d(j) = \min \{d(i) + c_{ij} : i \in N\}, \text{ for each } j \in N - \{s\}. \tag{3.2}$$

As usual, d(j) denotes the length of a shortest path from the source node to node j. These equations are known as *Bellman's equations* and represent necessary conditions for optimality of the shortest path problem. These conditions are also sufficient if every cycle in the network has a positive length. We will prove an alternate version of these conditions which is more suitable from the viewpoint of label correcting algorithms.

Theorem 3.2 Let d(i) for $i \in N$ be a set of labels. If d(s) = 0 and if in addition the labels satisfy the following conditions, then they represent the shortest path lengths from the source node:

C3.1. *d(i) is the length of some path from the source node to node i; and*
C3.2. $d(j) \le d(i) + c_{ij}$ *for all* $(i, j) \in A$.

Proof. Since d(i) is the length of some path from the source to node i, it is an upper bound on the shortest path length. We show that if the labels d(i) satisfy C3.2, then they are also lower bounds on the shortest path lengths, which implies the conclusion of the theorem. Consider any directed path P from the source to node j. Let P consist of nodes $s = i_1 - i_2 - i_3 - \ldots - i_k = j$. Condition C3.2 implies that $d(i_2) \le d(i_1) + c_{i_1 i_2} = c_{i_1 i_2}$, $d(i_3) \le d(i_2) + c_{i_2 i_3}, \ldots, d(i_k) \le d(i_{k-1}) + c_{i_{k-1} i_k}$. Adding these inequalities yields $d(j) = d(i_k) \le \sum_{(i,j) \in P} c_{ij}$. Therefore d(j) is a lower bound on the length of any directed path from the source to node j, including a shortest path from s to j. ■

We note that if the network contains a negative cycle then no set of labels d(i) satisfies C3.2. Suppose that the network did contain a negative cycle W and some labels d(i) satisfy C3.2. Consequently, $d(i) - d(j) + c_{ij} \ge 0$ for each $(i,j) \in W$. These inequalities imply that $\sum_{(i,j) \in W} (d(i) - d(j) + c_{ij}) = \sum_{(i,j) \in W} c_{ij} \ge 0$, since the labels d(i) cancel out in the summation. This conclusion contradicts our assumption that W is a negative cycle.

Conditions C3.1 in Theorem 3.2 correspond to primal feasibility for the linear programming formulation of the shortest path problem. Conditions C3.2 correspond to dual feasibility. From this perspective, we might view label correcting algorithms as methods that always maintain primal feasibility and try to achieve dual feasibility. The generic label correcting algorithm that we consider first is a general procedure for successively updating distance labels d(i) until they satisfy the conditions C3.2. At any point in the algorithm, the label d(i) is either ∞ indicating that we have yet to discover any path from the source to node j, or it is the length of some path from the source to node j. The algorithm is based upon the simple observation that whenever $d(j) > d(i) + c_{ij}$, the current path from the source to node i, of length d(i), together with the arc (i,j) is a shorter path to node j than the current path of length d(j).

```
algorithm LABEL CORRECTING;
begin
    d(s) : = 0 and pred(s) : = 0;
    d(j) : = ∞ for each j ∈ N – {s};
    while some arc (i,j) satisfies d(j) > d(i) + c_ij do
    begin
        d(j) : = d(i) + c_ij;
        pred(j) : = i;
    end;
end;
```

The correctness of the label correcting algorithm follows from Theorem 3.2. At termination, the labels d(i) satisfy $d(j) \le d(i) + c_{ij}$ for all $(i, j) \in A$, and hence represent the shortest path lengths. We now note that this algorithm is finite if there are no negative cost cycles and if the data are integral. Since d(j) is bounded from above by nC and below by -nC, the algorithm updates d(j) at most 2nC times. Thus when all data are integral, the number of distance updates is $O(n^2C)$, and hence the algorithm runs in pseudopolynomial time.

A nice feature of this label correcting algorithm is its flexibility: we can select the arcs that do not satisfy conditions C3.2 in any order and still assure the finite convergence. One drawback of the method, however, is that without a further restriction on the choice of arcs, the label correcting algorithm does not necessarily run in polynomial time. Indeed, if we start with pathological instances of the problem and make a poor choice of arcs at every iteration, then the number of steps can grow exponentially with n. (Since the algorithm is pseudopolynomial time, these instances do have exponentially large values of C.) To obtain a polynomial time bound for the algorithm, we can organize the computations carefully in the following manner. Arrange the arcs in A in some (possibly arbitrary) order. Now make passes through A. In each pass, scan arcs in A in order and check the condition $d(j) > d(i) + c_{ij}$; if the arc satisfies this condition, then update $d(j) = d(i) + c_{ij}$. Terminate the algorithm if no distance label changes during an entire pass. We call this algorithm the *modified label correcting algorithm.*

Theorem 3.3 *When applied to a network containing no negative cycles, the modified label correcting algorithm requires O(nm) time to determine shortest paths from the source to every other node.*

Proof. We show that the algorithm performs at most n–1 passes through the arc list. Since each pass requires O(1) computations for each arc, this conclusion implies the

O(nm) bound. Let $d^r(j)$ denote the length of the shortest path from the source to node j consisting of r or fewer arcs. Further, let $D^r(j)$ represent the distance label of node j after r passes through the arc list. We claim, inductively, that $D^r(j) \le d^r(j)$ for each $j \in N$, and for each $r = 1, \ldots, n-1$. We perform induction on the value of r. Suppose $D^{r-1}(j) \le d^{r-1}(j)$ for each $j \in N$. The provisions of the modified labeling algorithm imply that $D^r(j) \le \min \{ D^{r-1}(j), \min_{i \neq j} \{ D^{r-1}(i) + c_{ij} \} \}$. Next note that the shortest path to node j containing no more than r arcs either (i) has no more than r–1 arcs, or (ii) it contains exactly r arcs. In case (i), $d^r(j) = d^{r-1}(j)$, and in case (ii), $d^r(j) = \min_{i \neq j} \{d^{r-1}(i) + c_{ij}\}$. Consequently, $d^r(j) = \min \{d^{r-1}(j), \min_{i \neq j} \{d^{r-1}(i) + c_{ij}\}\} \ge \min \{D^{r-1}(j), \min_{i \neq j} \{D^{r-1}(i) + c_{ij}\}\}$; the inequality follows from the induction hypothesis. Hence, $D^r(j) \le d^r(j)$ for all $j \in N$. Finally, we note that the shortest path from the source to any node consists of at most n–1 arcs. Therefore, after at most n–1 passes, the algorithm terminates with the shortest path lengths. ■

The modified label correcting algorithm is also capable of detecting the presence of negative cycles in the network. If the algorithm does not update any distance label during an entire pass, up to the (n–1)-th pass, then it has a set of labels d(j) satisfying C3.2. In this case, the algorithm terminates with the shortest path distances and the network does not contain any negative cycle. On the other hand, when the algorithm modifies distance labels in all the n–1 passes, we make one more pass. If the distance label of some node i changes in the n-th pass, then the network contains a directed *walk* (a path together with a cycle that have one or more nodes in common) from node 1 to i of length greater than n–1 arcs that has smaller distance than all paths from the source node to i. This situation cannot occur unless the network contains a negative cost cycle.

Practical Improvements

As stated so far, the modified label correcting algorithm considers every arc of the network during every pass through the arc list. It need not do so. Suppose we order the arcs in the arc list by their tail nodes so that all arcs with the same tail node appear consecutively on the list. Thus, while scanning the arcs, we consider one node i at a time, scanning arcs in A(i) and testing the optimality conditions. Now suppose that during one pass through the arc list, the algorithm does not change the distance label of a node i. Then, during the next pass $d(j) \le d(i) + c_{ij}$ for every $(i, j) \in A(i)$ and the

algorithm need not test these conditions. To achieve this savings, the algorithm can maintain a list of nodes whose distance labels have changed since it last examined them. It scans this list in the first-in, first-out order to assure that it performs passes through the arc list A and, consequently, terminates in O(nm) time. The following procedure is a formal description of this further modification of the modified label correcting method.

```
algorithm MODIFIED LABEL CORRECTING;
begin
        d(s) : = 0 and pred(s) : = 0;
        d(j) : = ∞ for each j ∈ N – {s};
        LIST : = {s};
        while LIST ≠ ø do
        begin
                select the first element i of LIST;
                delete i from LIST;
                for each (i, j) ∈ A(i) do
                        if d(j) > d(i) + c_ij then
                        begin
                                d(j) : = d(i) + c_ij ;
                                pred(j) : = i;
                                if j ∉ LIST then add j to the end of LIST;
                        end;
        end;
end;
```

Another modification of this algorithm sacrifices its polynomial time behavior in the worst case, but greatly improves its running time in practice. The modification alters the manner in which the algorithm adds nodes to LIST. While adding a node i to LIST, we check to see whether the it has already appeared in the LIST. If yes, then we add i to the *beginning* of LIST, otherwise we add it to the *end* of LIST. This heuristic rule has the following plausible justification. If the node i has previously appeared on the LIST, then some nodes may have i as a predecessor. It is advantageous to update the distances for these nodes immediately, rather than update them from other nodes and then update them again when we consider node i. Empirical studies indicate that with this change alone, the algorithm is several times faster for many reasonable problem classes. Though this change makes the algorithm very attractive in practice, the worst-case

running time of the algorithm is exponential. Indeed, this version of the label correcting algorithm is the fastest algorithm in practice for finding the shortest path from a single source to all nodes in non-dense networks. (For the problem of finding a shortest path from a single source node to a single sink, certain variants of the label setting algorithm are more efficient in practice.)

3.5. All Pairs Shortest Path Algorithm

In certain applications of the shortest path problem, we need to determine shortest path distances between all pairs of nodes. In this section we describe two algorithms to solve this problem. The first algorithm is well suited for sparse graphs. It combines the modified label correcting algorithm and Dijkstra's algorithm. The second algorithm is better suited for dense graphs. It is based on dynamic programming.

If the network has nonnegative arc lengths, then we can solve the all pairs shortest path problem by applying Dijkstra's algorithm n times, considering each node as the source node once. If the network contains arcs with negative arc lengths, then we can first transform the network to one with nonnegative arc lengths as follows. Let s be a node from which all nodes in the network are reachable, i.e., connected by directed paths. We use the modified label correcting algorithm to compute shortest path distances from s to all other nodes. The algorithm either terminates with the shortest path distances d(j) or indicates the presence of a negative cycle. In the former case, we define the new length of the arc (i, j) as $\bar{c}_{ij} = c_{ij} + d(i) - d(j)$ for each $(i, j) \in A$. Condition C3.2 implies that $\bar{c}_{ij} \geq 0$ for all $(i, j) \in A$. Further, note that for any path P from node k to node l, $\sum_{(i, j) \in P} \bar{c}_{ij} = \sum_{(i, j) \in P} c_{ij} + d(k) - d(l)$ since the intermediate labels d(j) cancel out in the summation. This transformation thus changes the length of all paths between a pair of nodes by a constant amount (depending on the pair) and consequently preserves shortest paths. Since arc lengths become nonnegative after the transformation, we can apply Dijkstra's algorithm n-1 additional times to determine shortest path distances between all pairs of nodes in the transformed network. We then obtain the shortest path distance between nodes k and l in the original network by adding $d(l) - d(k)$ to the corresponding shortest path distance in the transformed network. This approach requires O(nm) time to solve the first shortest path problem, and if the network contains no negative cost cycle, the method takes an extra

O(n S(n, m, C) time to compute the remaining shortest path distances. In the expression S(n,m,C) is the time needed to solve a shortest path problem with nonnegative arc lengths. For the R-heap implementations of Dijkstra's algorithm we considered previously, S(n,m,C) = m + n log nC.

Another way to solve the all pairs shortest path problem is by dynamic programming. The approach we present is known as Floyd's algorithm. We define the variables $d^r(i, j)$ as follows:

$d^r(i, j)$ = the length of a shortest path from node i to node j subject to the condition that the path uses only the nodes 1, 2, ... , r–1 (and i and j) as internal nodes.

Let d(i, j) denote the actual shortest path distance. To compute $d^{r+1}(i, j)$, we first observe that a shortest path from node i to node j that passes through the nodes 1, 2, ... , r either (i) does not pass through the node r, in which case $d^{r+1}(i, j) = d^r(i, j)$, or (ii) does pass through the node r, in which case $d^{r+1}(i, j) = d^r(i, r) + d^r(r, j)$. Thus we have

$$d^1(i, j) = c_{ij},$$

and

$$d^{r+1}(i, j) = \min \{d^r(i, j), d^r(i, r) + d^r(r, j)\}.$$

We assume that $c_{ij} = \infty$ for all node pairs $(i, j) \notin A$. It is possible to solve the previous equations recursively for increasing values of r, and by varying the node pairs over N x N for a fixed value of r. The following procedure is a formal description of this algorithm.

```
algorithm  ALL PAIRS SHORTEST PATHS;
begin
    for all node pairs (i, j) ∈ N x N do d(i, j) : = ∞ and pred(i, j) : = 0;
    for each (i, j) ∈ A do d(i, j) : = c_ij and pred(i, j) : = i;
    for each r : = 1 to n do
            for each (i, j) ∈ N x N do
                if d(i, j) > d(i, r) + d(r, j) then
                begin
                        d(i, j) : = d(i, r) + d(r, j);
                        if i = j and d(i, i) < 0 then the network contains a negative cycle,
                            STOP;
                        pred(i, j) : = pred(r, j);
                end;
end;
```

Floyd's algorithm uses predecessor indices, *pred(i, j)*, for each node pair (i, j). The index pred(i, j) denotes the last node prior to node j in the tentative shortest path from node i to node j. The algorithm maintains the property that for each finite d(i, j), the network contains a path from node i to node j of length d(i, j). This path can be obtained by tracing the predecessor indices.

This algorithm performs n iterations, and in each iteration it performs O(1) computations for each node pair. Consequently, it runs in $O(n^3)$ time. The algorithm either terminates with the shortest path distances or stops when d(i, i) < 0 for some node i. In the latter case, for some node $r \neq i$, d(i, r) + d(r, i) < 0. Hence, the union of the tentative shortest paths from node i to node r and from node r to node i contains a negative cycle. This cycle can be obtained by using the predecessor indices.

Floyd's algorithm is in many respects similar to the modified label correcting algorithm. This relationship becomes more transparent from the following theorem.

Theorem 3.4 *If d(i, j) for (i, j) ∈ N x N satisfy the following conditions, then they represent the shortest path distances:*

(i) $d(i, i) = 0$ *for all i;*
(ii) *d(i, j) is the length of some path from node i to node j;*
(iii) $d(i, j) \leq d(i, r) + c_{rj}$ *for all i, r, and j.*

Proof. For fixed i, this theorem is a consequence of Theorem 3.2. ■

4. MAXIMUM FLOWS

An important characteristic of a network is its capacity to carry flow. What, given capacities on the arcs, is the maximum flow that can be sent between any two nodes? The resolution of this question determines the "best" use of arc capacities and establishes a reference point against which to compare other ways of using the network. Moreover, the solution of the maximum flow problem with capacity data chosen judiciously establishes other performance measures for a network. For example, what is the minimum number of nodes whose removal from the network destroys all paths joining a particular pair of nodes? Or, what is the maximum number of node disjoint paths that join this pair of nodes? These and similar reliability measures indicate the robustness of the network to failure of its components.

In this section, we discuss several algorithms for computing the maximum flow between two nodes in a network. We begin by introducing a basic labeling algorithm for solving the maximum flow problem. The validity of these algorithms rests upon the celebrated *max-flow min-cut theorem* of network flows. This remarkable theorem has a number of surprising implications in machine and vehicle scheduling, communication systems planning and several other application domains. We then consider improved versions of the basic labeling algorithm with better theoretical performance guarantees. In particular, we describe *preflow push* algorithms that have recently emerged as the most powerful techniques for solving the maximum flow problem, both theoretically and computationally.

We consider a capacitated network G = (N, A) with a *nonnegative integer* capacity u_{ij} for any arc (i, j) $\in$ A. The source s and sink t are two distinguished nodes of the network. We assume that for every arc (i, j) in A, (j, i) is also in A. There is no loss of generality in making this assumption since we allow zero capacity arcs. We also assume without any loss of generality that all arc capacities are finite (since we can set the capacity of any uncapacitated arc equal to the sum of the capacities of all capacitated arcs). Let $U = \max \{u_{ij} : (i, j) \in A\}$. As earlier, the arc adjacency list defined as A(i) = {(i, k) : (i, k) $\in$ A} designates the arcs emanating from node i. In the maximum flow problem, we wish to find the maximum flow from the source node s to the sink node t that satisfies the arc capacities. Formally, the problem is to

Maximize v (4.1a)

subject to

$$\sum_{\{j\,:\,(i,\,j)\,\in\,A\}} x_{ij} \;-\; \sum_{\{j\,:\,(j,\,i)\,\in\,A\}} x_{ji} = \begin{cases} v, \text{ if } i = s, \\ 0, \text{ if } i \neq s, t, \text{ for all } i \in N, \\ -v, \text{ if } i = t, \end{cases} \qquad (4.1b)$$

$$0 \le x_{ij} \le u_{ij}, \text{ for each } (i, j) \in A. \qquad (4.1c)$$

It is possible to relax the integrality assumption on arc capacities for some algorithms, though this assumption is necessary for others. Algorithms whose complexity bounds involve U assume integrality of data. Note, however, that rational arc capacities can always be transformed to integer arc capacities by appropriately scaling the data. Thus, the integrality assumption is not a restrictive assumption in practice.

The concept of *residual network* is crucial to the algorithms we consider. Given a flow x, the *residual capacity*, r_{ij}, of any arc $(i, j) \in A$ represents the maximum additional flow that can be sent from node i to node j using the arcs (i, j) and (j, i). The residual capacity has two components: (i) $u_{ij} - x_{ij}$, the unused capacity of arc (i, j), and (ii) the current flow x_{ji} on arc (j, i) which can be cancelled to increase flow to node j. Consequently, $r_{ij} = u_{ij} - x_{ij} + x_{ji}$. We call the network consisting of the arcs with positive residual capacities the *residual network* (with respect to the flow x), and represent it as G(x). Figure 4.1 illustrates an example of a residual network.

4.1 Labeling Algorithm and the Max-Flow Min-Cut Theorem

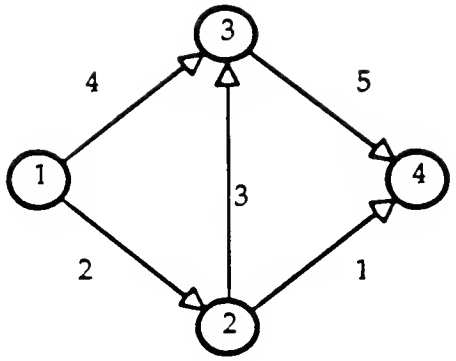
One of the simplest and most intuitive algorithms for solving the maximum flow problem is the *augmenting path algorithm* due to Ford and Fulkerson. The algorithm proceeds by identifying directed paths from the source to the sink in the residual network and augmenting flows on these paths, until the residual network contains no such path. The following high-level (and flexible) description of the algorithm summarizes the basic iterative steps, without specifying any particular algorithmic strategy for how to determine augmenting paths.

```
algorithm AUGMENTING PATH;
begin
    x : = 0;
    while there is a path P from s to t in G(x) do
    begin
        Δ : = min {r_ij : (i, j) ∈ P};
        augment Δ units of flow along P and update G(x);
    end;
end;
```

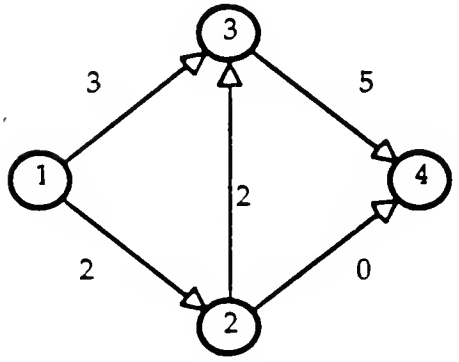
For each (i, j) $\in$ P, augmenting Δ units of flow along P decreases r_{ij} by Δ and increases r_{ji} by Δ. We now discuss this algorithm in more detail. First, we need a method to identify a directed path from the source to the sink in the residual network or to show that the network contains no such path. Second, we need to show that the algorithm terminates finitely. Finally, we must establish that the algorithm terminates with a maximum flow. The last result follows from the proof of the *max-flow min-cut theorem*.

A directed path from the source to the sink in the residual network is also called an *augmenting path*. The residual capacity of an augmenting path is the minimum residual capacity of any arc on the path. The definition of the residual capacity implies that an additional flow of Δ in arc (i, j) of the residual network corresponds to (i) an increase in x_{ij} by Δ in the original network, or (ii) a decrease in x_{ji} by Δ in the original network, or (iii) a convex combination of (i) and (ii). For our purposes, it is easier to work directly with residual capacities and to compute the flows only when the algorithm terminates.

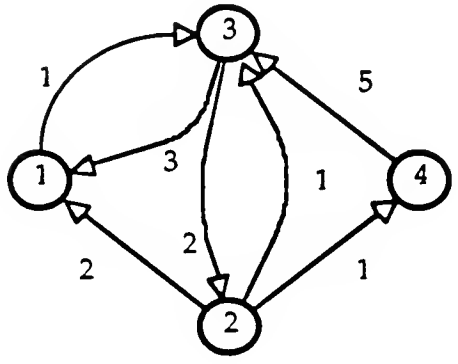
The labeling algorithm performs a search of the residual network to find a directed path from s to t. It does so by *fanning out* from the source node s to find a directed tree containing nodes that are reachable from the source along a directed path in the residual network. At any step, we refer to the nodes in the tree as *labeled* and those not in the tree as *unlabeled*. The algorithm selects a labeled node and scans its arc adjacency list (in the residual network) to label more unlabeled nodes. Eventually, the sink becomes labeled and the algorithm sends the maximum possible flow on the path from s to t. It then erases the labels and repeats this process. The algorithm terminates when it has scanned all labeled nodes and the sink remains unlabeled. The following algorithmic description specifies the steps of the labeling algorithm in detail. The



a. Network with arc capacities.
Node 1 is the source and node 4 is the sink.
(Arcs not shown have zero capacities.)



b. Network with a flow x.



c. The residual network with residual arc capacities.

Figure 4.1 Example of a residual network.

algorithm maintains a predecessor index, *pred(i)*, for each labeled node i indicating the node that caused node i to be labeled. The predecessor indices allow us to trace back along the path from a node to the source.

```
algorithm LABELING;
begin
    loop
        pred(j) : = 0  for  each j ∈ N;
        L : = {s};
        while L ≠ ø and t is unlabeled do
        begin
            select a node i ∈ L;
            for each (i, j) ∈ A(i) do
                if j is unlabeled and r_ij > 0 then
                begin
                    pred(j) : = i;
                    mark j as labeled and add this node to L;
                end
        end;

        if  t  is labeled then
        begin
            use the predecessor labels to trace back to obtain the augmenting path P
                from s to t;
            Δ : = min {r_ij : (i, j) ∈ P};
            augment  Δ units of flow along P;
            erase all labels and go to loop;
        end
        else  quit the loop;
    end; {loop}
end;
```

The final residual capacities r can be used to obtain the arc flows as follows. Since $r_{ij} = u_{ij} - x_{ij} + x_{ji}$, the arc flows satisfy $x_{ij} - x_{ji} = u_{ij} - r_{ij}$. Hence, if $u_{ij} > r_{ij}$ we can set $x_{ij} = u_{ij} - r_{ij}$ and $x_{ji} = 0$; otherwise we set $x_{ij} = 0$ and $x_{ji} = r_{ij} - u_{ij}$.

In order to show that the algorithm obtains a maximum flow, we introduce some new definitions and notation. Recall from Section 1.3 that a set $Q \subseteq A$ is a *cutset* if the subnetwork $G' = (N, A - Q)$ is disconnected and no superset of Q has this property. A cutset partitions set N into two subsets. A cutset is called an *s-t cutset* if the source and the sink nodes are contained in different subsets of nodes S and $\bar{S} = N - S$: S is the set of nodes connected to s. Conversely, any partition of the node set as S and $\bar{S}$ with $s \in S$ and $t \in \bar{S}$ defines an s-t cutset. Consequently, we alternatively designate an s-t cutset as $(S, \bar{S})$. An arc (i, j) with $i \in S$ and $j \in \bar{S}$ is called a *forward arc*, and an arc (i, j) with $i \in \bar{S}$ and $j \in S$ is called a *backward arc* in the cutset $(S, \bar{S})$.

Let x be a flow vector satisfying the flow conservation and capacity constraints of (4.1). For this flow vector x, let v be the amount of flow leaving the source. We refer to v as the *value* of the flow. The flow x determines the net flow across an s-t cutset $(S, \bar{S})$ as

$$F_x(S, \bar{S}) = \sum_{i \in S} \sum_{j \in \bar{S}} x_{ij} - \sum_{i \in \bar{S}} \sum_{j \in S} x_{ij}. \tag{4.2}$$

Define the capacity $C(S, \bar{S})$ of an s-t cutset $(S, \bar{S})$ is defined as

$$C(S, \bar{S}) = \sum_{i \in S} \sum_{j \in \bar{S}} u_{ij}. \tag{4.3}$$

We claim that the flow across any s-t cutset equals the value of the flow and does not exceed the cutset capacity. Adding the flow conservation constraints (4.1 b) for nodes in S and noting that when nodes i and j both belong to S, x_{ij} in equation for node j cancels $-x_{ij}$ in equation for node i, we obtain

$$v = \sum_{i \in S} \sum_{j \in \bar{S}} x_{ij} - \sum_{i \in \bar{S}} \sum_{j \in S} x_{ij} \equiv F_x(S, \bar{S}). \tag{4.4}$$

Substituting $x_{ij} \leq u_{ij}$ in the first summation and $x_{ij} \geq 0$ in the second summation shows that

$$F_x(S, \bar{S}) \leq \sum_{i \in S} \sum_{j \in \bar{S}} u_{ij} \equiv C(S, \bar{S}). \tag{4.5}$$

This result is the weak duality property of the maximum flow problem when viewed as a linear program. Like most weak duality results, it is the "easy" half of the duality theory. The more substantive strong duality property asserts that (4.5) holds as an equality for some choice of x and some choice of an s-t cutset $(S, \bar{S})$. This strong duality property is the max-flow min-cut theorem.

Theorem 4.1. (Max-Flow Min-Cut Theorem) The maximum value of flow from s to t equals the minimum capacity of all s-t cuts.

Proof. Let x denote the a maximum flow vector and v denote the maximum flow value. (Linear programming theory, or our subsequent algorithmic developments, guarantee that the problem always has a maximum flow as long as some cutset has finite capacity.) Define S to be the set of labeled nodes in the residual network G(x) when we apply the labeling algorithm with the initial flow x. Let $\bar{S} = N - S$. Clearly, since x is a maximum flow, $s \in S$ and $t \in \bar{S}$. Adding the flow conservation equations for nodes in S, we obtain (4.4). Note that nodes in $\bar{S}$ cannot be labeled from the nodes in S, hence $r_{ij} = 0$ for each forward arc (i, j) in the cutset $(S, \bar{S})$. Since $r_{ij} = u_{ij} - x_{ij} + x_{ji}$, the conditions $x_{ij} \le u_{ij}$ and $x_{ji} \ge 0$ imply that $x_{ij} = u_{ij}$ for each forward arc in the cutset $(S, \bar{S})$ and $x_{ij} = 0$ for each backward arc in the cutset. Making these substitutions in (4.4) yields

$$v = F_x(S, \bar{S}) = \sum_{i \in S} \sum_{j \in \bar{S}} u_{ij} = C(S, \bar{S}). \qquad (4.6)$$

But we have observed earlier that v is a lower bound on the capacity of any s-t cutset. Consequently, the cutset $(S, \bar{S})$ is a minimum capacity s-t cutset and its capacity equals the maximum flow value v. We thus have established the theorem. ■

The proof of this theorem not only establishes the max-flow min-cut property, but the same argument shows that when the labeling algorithm terminates, it has at hand both the maximum flow value (and a maximum flow vector) and a minimum capacity s-t cutset. But does it terminate finitely? Each labeling iteration of the algorithm scans any node at most once, inspecting each arc in A(i). Consequently, the labeling iteration scans each arc at most once and requires O(m) computations. If all arc capacities are integral and bounded by a finite number U, then the capacity of the cutset $(s, N - \{s\})$ is at most nU. Since the labeling algorithm increases the flow value by at least one unit in any iteration, it terminates within nU iterations. This bound on the number of iterations is not entirely satisfactory for large values of U; if $U = 2^n$, the bound is

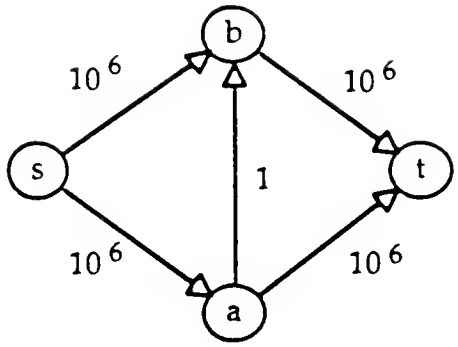
exponential in the number of nodes. Moreover, the algorithm can indeed perform that many iterations. In addition, if the capacities are irrational, the algorithm may not terminate: although the successive flow values converge, they may not converge to the maximum flow value. Thus if the method is to be effective, we must select the augmenting paths carefully. Several refinements of the algorithms, including those we consider in Section 4.2 – 4.4 overcome this difficulty and obtain an optimum flow even if the capacities are irrational; moreover, the max-flow min-cut theorem (and our proof of Theorem 4.1) is true even if the data are irrational.

A second drawback of the labeling algorithm is its *"forgetfulness"*. At each iteration, the algorithm generates node labels that contain information about augmenting paths from the source to other nodes. The implementation we have described erases the labels when it proceeds from one iteration to the next, even though much of this information may be valid in the next residual network. Erasing the labels therefore destroys potentially useful information. Ideally, we should retain a label when it can be used profitably in later computations.

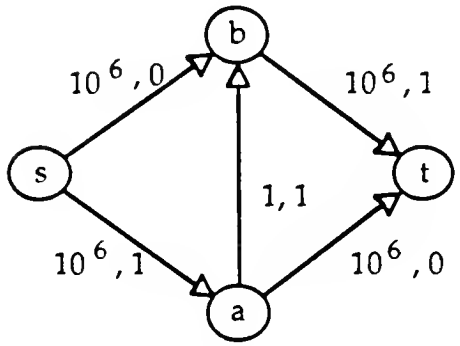
4.2 Decreasing the Number of Augmentations

The bound of nU on the number of augmentations in the labeling algorithm is not satisfactory from a theoretical perspective. Furthermore, without further modifications, the augmenting path algorithm may take $\Omega(nU)$ augmentations, as the example given in Figure 4.2 illustrates.

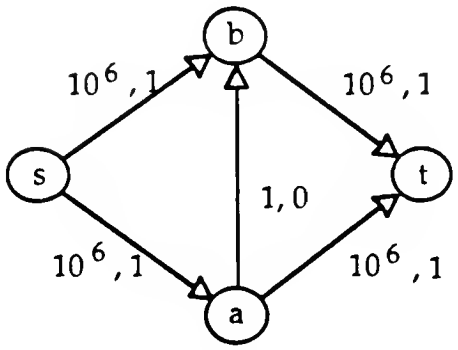
Flow decomposition shows that, in principle, augmenting path algorithms should be able to find a maximum flow in no more than m augmentations. For suppose x is an optimum flow and y is any initial flow (possibly zero). By the flow decomposition property, it is possible to obtain x from y by a sequence of at most m augmentations on augmenting paths from s to t plus flows around augmenting cycles. If we define x' as the flow vector obtained from y by applying only the augmenting paths, then x' also is a maximum flow (flows around cycles do not change flow value). This result shows that it is, in theory, possible to find a maximum flow using at most m augmentations. Unfortunately, to apply this flow decomposition argument, we need to know a maximum flow. No algorithm developed in the literature comes close to achieving this theoretical bound. Nevertheless, it is possible to improve considerably on the bound of O(nU) augmentations of the basic labeling algorithm.



(a)



(b)



(c)

Figure 4.2 A pathological example for the labeling algorithm.

(a) The input network with arc capacities.

(b) After augmenting along the path s-a-b-t. Arc flow is indicated beside the arc capacity.

(c) After augmenting along the path s-b-a-t. After 2×10^6 augmentations, alternately along s-a-b-t and s-b-a-t, the flow is maximum.

One natural specialization of the augmenting path algorithm is to augment flow along a "shortest path" from the source to the sink, defined as a path consisting of the least number of arcs. If we augment flow along a shortest path, then the length of any shortest path either stays the same or increases. Moreover, within m augmentations, the length of the shortest path is guaranteed to increase. (We will prove these results in the next section.) Since no path contains more than n-1 arcs, this rule guarantees that the number of augmentations is at most (n-1)m.

An alternative is to augment flow along a path of maximum residual capacity. This specialization also leads to improved complexity. Let v be any flow value and v^* be the maximum flow value. By flow decomposition, the network contains at most m augmenting paths whose residual capacities sum to $(v^* - v)$. Thus the maximum capacity augmenting path has residual capacity at least $(v^* - v)/m$. Now consider a sequence of 2m consecutive maximum capacity augmentations, starting with flow v. At least one of these augmentations must augment the flow by an amount $(v^* - v)/2m$ or less, for otherwise we will have a maximum flow. Thus after 2m or fewer augmentations, the algorithm would reduce the capacity of a maximum capacity augmenting path by a factor of at least two. Since this capacity is initially at most U and the capacity must be at least 1 until the flow is maximum, after O(m log U) maximum capacity augmentations, the flow must be maximum. (Note that we are essentially repeating the argument used to establish the geometric improvement approach discussed in Section 1.6.)

In the following section, we consider another algorithm for reducing the number of augmentations.

4.3 Shortest Augmenting Path Algorithm

A natural approach to augmenting along shortest paths would be to successively look for shortest paths by performing a breadth first search in the residual network. If the labeling algorithm maintains the set L of labeled nodes as a queue, then by examining the labeled nodes in a first-in, first-out order, it would obtain a shortest path in the residual network. Each of these iterations would take O(m) steps both in the worst case and in practice, and (by our subsequent observations) the resulting computation time would be $O(nm^2)$. Unfortunately, this computation time is excessive. We can improve this running time by exploiting the fact that the minimum distance from any

node i to the sink node t is monotonically nondecreasing over all augmentations. By fully exploiting this property, we can reduce the average time per augmentation to O(n).

The Algorithm

The concept of distance labels will prove to be an important construct in the maximum flow algorithms that we discuss in this section and in Sections 4.4 and 4.5. A *distance function* $d : N \rightarrow Z^+$ with respect to the residual capacities r_{ij} is a function from the set of nodes to the nonnegative integers. We say that a distance function is *valid* if it satisfies the following two conditions:

C4.1. $d(t) = 0$;
C4.2. $d(i) \leq d(j) + 1$ for every arc $(i, j) \in A$ with $r_{ij} > 0$.

We refer to d(i) as the *distance label* of node i and condition C4.2 as the *validity condition*. It is easy to demonstrate that d(i) is a lower bound on the length of the shortest directed path from i to t in the residual network. Let $i = i_1 - i_2 - i_3 - \ldots - i_k - t$ be any path of length k in the residual network from node i to t. Then, from C4.2 we have $d(i) = d(i_1) \leq d(i_2) + 1$, $d(i_2) \leq d(i_3) + 1, \ldots, d(i_k) \leq d(t) + 1 = 1$. These inequalities imply that $d(i) \leq k$ for *any* path of length k in the residual network and, hence, any shortest path from node i to t contains at least d(i) arcs. If for each node i, the distance label d(i) equals the length of the shortest path from i to t in the residual network, then we call the distance labels *exact*. For example, in Figure 4.1(c), d = (0, 0, 0, 0) is a valid distance label, though d = (3, 1, 2, 0) represents the exact distance label.

We now define some additional notation. An arc (i, j) in the residual network is *admissible* if it satisfies $d(i) = d(j) + 1$. Other arcs are *inadmissible*. A path from s to t consisting entirely of admissible arcs is an *admissible path*. The algorithm we describe next repeatedly augments flow along admissible paths. For any admissible path of length k, d(s) = k. Since d(s) is a lower bound on the length of any path from the source to the sink, the algorithm augments flows along shortest paths in the residual network. Thus, we refer to the algorithm as the *shortest augmenting path algorithm*.

Whenever we augment along a path, each of the distance labels for nodes in the path is exact. However, for other nodes in the network it is not necessary to maintain exact distances; it suffices to have valid distances, which are lower bounds on the exact distances. There is no particular urgency to compute these distances exactly. By allowing the distance label of node i to be less than the distance from i to t, we maintain flexibility in the algorithm, without incuring any significant cost.

We can compute the initial distance labels by performing a backward breadth first search of the residual network, starting at the sink node. The algorithm generates an admissible path by adding admissible arcs, one at a time, as follows. It maintains a path from the source node to some node i*, called the *current node*, consisting entirely of admissible arcs. We call this path a *partial admissible path* and store it using *predecessor* indices, i.e., pred(j) = i for each arc (i, j) on the path. The algorithm performs one of the two steps at the current node: *advance* or *retreat*. The advance step identifies some admissible arc (i*, j*) emanating from node i*, adds it to the partial admissible path, and designates j* as the new current node. If no admissible arc emanates from node i*, then the algorithm performs the retreat step. This step increases the distance label of node i* so that at least one admissible arc emanates from it (we refer to this step as a *relabel* operation). Increasing d(i*) makes the arc (pred(i*), i*) inadmissible (assuming $i^* \neq s$). Consequently, we delete (pred(i*), i*) from the partial admissible path and node pred(i*) becomes the new current node. Whenever the partial admissible path becomes an admissible path (i.e., contains node t), the algorithm makes a maximum possible augmentation on this path and begins again with the source as the current node. The algorithm terminates when $d(s) \geq n$, indicating that the network contains no augmenting path from the source to the sink. We next describe the algorithm formally.

```
algorithm SHORTEST AUGMENTING PATH;
begin
        x : = 0;
        perform backward breadth first search of the residual network
            from node t to obtain the distance labels d(i);
        i* : = s;
        while d(s) < n do
        begin
                if i* has an admissible arc then ADVANCE(i*)
                else RETREAT(i*);
                if i* = t then AUGMENT and set i* : = s;
        end;
end;

procedure ADVANCE(i*);
begin
        let (i*, j*) be an admissible arc in A(i*);
        pred(j*) : = i* and i* : = j*;
end;
```

```
procedure RETREAT(i*);
begin
    d(i*) : = min { d(j) + 1 : (i, j) ∈ A(i*) and r_ij > 0 };
    if i* ≠ s then i* : = pred(i*);
end;

procedure AUGMENT;
begin
    using predecessor indices identify an augmenting path P from the source to the
        sink;
    Δ : = min {r_ij : (i, j) ∈ P};
    augment Δ units of flow along path P;
end;
```

We use the following data structure to select an admissible arc emanating from a node. We maintain the list A(i) of arcs emanating from each node i. Arcs in each list can be arranged arbitrarily, but the order, once decided, remains unchanged throughout the algorithm. Each node i has a *current-arc* (i, j) which is the current candidate for the next advance step. Initially, the current-arc of node i is the first arc in its arc list. The algorithm examines this list sequentially and whenever the current arc is inadmissible, it makes the next arc in the arc list the current arc. When the algorithm has examined all arcs in A(i), it updates the distance label of node i and the current arc once again becomes the first arc in its arc list. In our subsequent discussion we shall always implicitly assume that the algorithms select admissible arcs using this technique.

Correctness of the Algorithm

We first show that the shortest augmentation algorithm correctly solves the maximum flow problem.

Lemma 4.1. *The shortest augmenting path algorithm maintains valid distance labels at each step. Moreover, each relabel step strictly increases the distance label of a node.*

Proof. We show that the algorithm maintains valid distance labels at every step by performing induction on the number of augment and relabel steps. Initially, the algorithm constructs valid distance labels. Assume, inductively, that the distance function is valid prior to a step, i.e., satisfies the validity condition C4.2. We need to check whether these conditions remain valid (i) after an augment step (when the residual graph changes), and (ii) after a relabel step.

(i) A flow augmentation on arc (i, j) might delete this arc from the residual network, but this modification to the residual network does not affect the validity of the distance function for this arc. Augmentation on arc (i, j) might, however, create an additional arc (j, i) with $r_{ji} > 0$ and, therefore, also create an additional condition $d(j) \leq d(i) + 1$ that needs to be satisfied. The distance labels satisfy this validity condition, though, since $d(i) = d(j) + 1$ by the admissibility property of the augmenting path.

(ii) The algorithm performs a relabel step at node i when the current arc reaches the end of arc list A(i). Observe that if an arc (i, j) is inadmissible at some stage, then it remains inadmissible until d(i) increases because of our inductive hypothesis that distance labels are nondecreasing. Thus, when the current arc reaches the end of the arc list A(i), then no arc $(i, j) \in A(i)$ satisfies $d(i) = d(j) + 1$ and $r_{ij} > 0$. Hence, $d(i) < \min\{d(j) + 1 : (i, j) \in A(i) \text{ and } r_{ij} > 0\} = d'(i)$, thereby establishing the second part of the lemma.

Finally, the choice for changing d(i) ensures that the condition $d(i) \leq d(j) + 1$ remains valid for all (i, j) in the residual network; in addition, since d(i) increases, the conditions $d(k) \leq d(i) + 1$ remain valid for all arcs (k, i) in the residual network. ■

Theorem 4.2. The shortest augmenting path algorithm correctly computes a maximum flow.

Proof. The algorithm terminates when $d(s) \geq n$. Since d(s) is a lower bound on the length of the shortest augmenting path from s to t, this condition implies that the network contains no augmenting path from the source to the sink, which is the termination criterion for the generic augmenting path algorithm. ■

At termination of the algorithm, we can obtain a minimum s-t cutset as follows. For $0 \leq k \leq n$, let α_k denote the number of nodes with distance label equal to k. Note that α_{k^*} must be zero for some $k^* \leq n - 1$ since $\sum_{k=0}^{n-1} \alpha_k \leq n - 1$. (Recall that $d(s) \geq n$.) Let $S = \{i \in N: d(i) > k^*\}$ and $\bar{S} = N - S$. When $d(s) \geq n$ and the algorithm terminates, $s \in S$ and $t \in \bar{S}$, and both the sets S and $\bar{S}$ are nonempty. Consider the s-t cutset $(S, \bar{S})$. By construction, $d(i) > d(j) + 1$ for all $(i, j) \in (S, \bar{S})$. The validity condition C4.2 implies that $r_{ij} = 0$ for each $(i, j) \in (S, \bar{S})$. Hence, $(S, \bar{S})$ is a minimum s-t cutset and the current flow is maximum.

Complexity of the Algorithm

We next show that the algorithm computes a maximum flow in $O(n^2m)$ time.

Lemma 4.2. (a) Each distance label increases at most n times. Consequently, the total number of relabel steps is at most n^2. (b) The number of augment steps is at most nm/2.

Proof. Each relabel step at node i increases d(i) by at least one. After the algorithm has relabeled node i at most n times, $d(i) \geq n$. From this point on, the algorithm never selects node i again during an advance step since for every node k in the current path, $d(k) < d(s) < n$. Thus the algorithm relabels a node at most n times and the total number of relabel steps is bounded by n^2.

Each augment step saturates at least one arc, i.e., decreases its residual capacity to zero. Suppose that the arc (i, j) becomes saturated at some iteration (at which $d(i) = d(j) + 1$). Then no more flow can be sent on (i, j) until flow is sent back from j to i (at which point $d'(j) = d'(i) + 1 \geq d(i) + 1 = d(j) + 2$). Hence, between two consecutive saturations of arc (i, j), d(j) increases by at least 2 units. Consequently, any arc (i, j) can become saturated at most n/2 times and the total number of arc saturations is no more than nm/2. ■

Theorem 4.3. The shortest augmenting path algorithm runs in $O(n^2m)$ time.

Proof. The algorithm performs O(nm) flow augmentations and each augmentation takes O(n) time, resulting in $O(n^2m)$ total effort in the augmentation steps. Each advance step increases the length of the partial admissible path by one, and each retreat step decreases its length by one; since each partial admissible path has length at most n, the algorithm requires at most $O(n^2 + n^2m)$ advance steps. The first term comes from the number of retreat (relabel) steps, and the second term from the number of augmentations, which are bounded by nm/2 by the previous lemma.

For each node i, the algorithm performs the relabel operation O(n) times, each execution requiring $O(|A(i)|)$ time. The total time spent in all relabel operations is $\sum_{i \in N} n|A(i)| = O(nm)$. Finally, we consider the time spent in identifying admissible arcs. The time taken to identify the admissible arc of node i is O(1) plus the time spent in scanning arcs in A(i). After having performed $|A(i)|$ such scannings, the algorithm reaches the end of the arc list and relabels node i. Thus the total time spent in all

scannings is $O(\sum_{i \in N} n|A(i)|) = O(nm)$. The combination of these time bounds establishes the theorem. ■

The proof of Theorem 4.3 also suggests an alternative termination condition for the shortest augmenting path algorithm. The termination criteria of $d(s) \geq n$ is satisfactory for a worst-case analysis, but may not be efficient in practice. Researchers have observed empirically that the algorithm spends too much time in relabeling, a major portion of which is done *after* it has already found the maximum flow. The algorithm can be improved by detecting the presence of a minimum cutset prior to performing these relabeling operations. We can do so by maintaining the number of nodes α_k with distance label equal to k, for $0 \leq k \leq n$. The algorithm updates this array after every relabel operation and terminates whenever it first finds a *gap* in the α array, i.e., $\alpha_{k^*} = 0$ for some $k^* < n$. As we have seen earlier, if $S = \{ i : d(s) > k^* \}$, then $(S, \bar{S})$ denotes a minimum cutset.

The idea of augmenting flows along shortest paths is intuitively appealing and easy to implement in practice. The resulting algorithms identify at most O(nm) augmenting paths and this bound is tight, i.e., on particular examples these algorithms perform $\Omega(nm)$ augmentations. The only way to improve the running time of the shortest augmenting path algorithm is to perform fewer computations per augmentation. The use of a sophisticated data structure, called *dynamic trees* , reduces the average time for each augmentation from O(n) to O(log n). This implementation of the maximum flow algorithm runs in O(nm log n) time and obtaining further improvements appears quite difficult except in very dense networks. These implementations with sophisticated data structures appear to be primarily of theoretical interest, however, because maintaining the data structures requires substantial overhead that tends to increase rather than reduce the computational times in practice. A detailed discussion of dynamic trees is beyond the scope of this chapter.

Potential Functions and an Alternate Proof of Lemma 4.2(b)

A powerful method for proving computational time bounds is to use *potential functions*. Potential function techniques are general purpose techniques for proving the complexity of an algorithm by analyzing the effects of different steps on an appropriately defined function. The use of potential functions enables us to define an "accounting" relationship between the occurrences of various steps of an algorithm that can be used to

obtain a bound on the steps that might be difficult to obtain using other arguments. Rather than formally introducing potential functions, we illustrate the technique by showing that the number of augmentations in the shortest augmenting path algorithm is O(nm).

Suppose in the shortest augmenting path algorithm we kept track of the number of admissible arcs in the residual network. Let F(k) denote the number of admissible arcs at the end of the k-th step; for the purpose of this argument, we count a step either as an augmentation or as a relabel operation. Let the algorithm perform K steps before it terminates. Clearly, $F(0) \leq m$ and $F(K) \geq 0$. Each augmentation decreases the residual capacity of at least one arc to zero and hence reduces F by at least one unit. Each relabeling of node i creates as many as $|A(i)|$ new admissible arcs, and increases F by the same amount. This increase in F is at most nm over all relabelings, since the algorithm relabels any node at most n times (as a consequence of Lemma 4.1) and $\sum_{i \in N} n|A(i)| = nm$. Since the initial value of F is at most m more than its terminal value, the total decrease in F due to all augmentations is m + nm. Thus the number of augmentations is at most m + nm = O(nm).

This argument is fairly representative of the potential function argument. Our objective was to bound the number of augmentations. We did so by defining a potential function that decreases whenever the algorithm performs an augmentation. The potential increases only when the algorithm relabels distances, and thus we can bound the number of augmentations using bounds on the number of relabels. In general, we bound the number of steps of one type in terms of known bounds on the number of steps of other types.

4.4 Preflow-Push Algorithms

Augmenting path algorithms send flow by augmenting along a path. This basic step further decomposes into the more elementary operation of sending flow along an arc. Thus sending a flow of Δ units along a path of k arcs decomposes into k basic operations of sending a flow of Δ units along an arc of the path. We shall refer to each of these basic operations as a *push*.

A path augmentation has one advantage over a single push: it maintains conservation of flow at all nodes. In fact, the push-based algorithms such as those we develop in this and the following sections necessarily violate conservation of flow.

Rather, these algorithms permit the flow into a node to exceed the flow out of this node. We will refer to any such flows as *preflows*. The two basic operations of the generic preflow-push methods are (i) pushing the flow on an admissible arc, and (ii) updating a distance label, as in the augmenting path algorithm described in the last section. (We define the distance labels and admissible arcs as in the previous section.)

Preflow-push algorithms have several advantages over augmentation based algorithms. First, they are more general and more flexible. Second, they can push flow closer to the sink before identifying augmenting paths. Third, they are better suited for distributed or parallel computation. Fourth, the best preflow-push algorithms currently outperform the best augmenting path algorithms in theory as well as in practice.

The Generic Algorithm

A *preflow* x is a function x: A → R that satisfies (4.1c) and the following relaxation of (4.1b):

$$\sum_{\{j\,:\,(j,\,i)\,\in\,A\}} x_{ji} \;-\; \sum_{\{j\,:\,(i,\,j)\,\in\,A\}} x_{ij} \;\geq 0 \;, \text{for all } i \in N - \{s, t\}.$$

The preflow-push algorithms maintain a preflow at each intermediate stage. For a given preflow x, we define the *excess* for each node $i \in N - \{s, t\}$ as

$$e(i) = \sum_{\{j\,:\,(j,\,i)\,\in\,A\}} x_{ji} \;-\; \sum_{\{j\,:\,(i,\,j)\,\in\,A\}} x_{ij} \;.$$

We refer to a node with positive excess as an *active* node. We adopt the convention that the source and sink nodes are never active. The preflow-push algorithms perform all operations using only local information. At each iteration of the algorithm (except its initialization and its termination), the network contains at least one active node, i.e., a node $i \in N - \{s, t\}$ with $e(i) > 0$. The goal of each iterative step is to choose some active node and to send its excess *closer* to the sink, closeness being measured with respect to the current distance labels. As in the shortest augmenting path algorithms, we send flow only on admissible arcs. If the method cannot send excess from this node to nodes with smaller distance labels, then it increases the distance label of the node so that it creates at least one new admissible arc. The algorithm terminates when the network contains no active nodes. The preflow-push algorithm uses the following subroutines:

```
procedure PREPROCESS;
begin
        x := 0;
        perform a backward breadth first-search of the residual network, starting at node t,
              to determine initial distance labels d(i);
        x_sj := u_sj for each arc (s, j) ∈ A(s) and d(s) := n;
end;

procedure PUSH/RELABEL(i);
begin
        if the network contains an admissible arc (i, j) then
                push δ := min{e(i), r_ij} units of flow from node i to node j
        else replace d(i) by min {d(j) + 1 : (i, j) ∈ A(i) and r_ij > 0};
end;
```

A push of δ units from node i to node j decreases both e(i) and r_{ij} by δ units and increases both e(j) and r_{ji} by δ units. We say that a push of δ units of flow on arc (i, j) is *saturating* if $\delta = r_{ij}$ and *nonsaturating* otherwise. We refer to the process of increasing the distance label of a node as a *relabel* operation. The purpose of the relabel operation is to create at least one admissible arc on which the algorithm can perform further pushes.

The following generic version of the preflow-push algorithm combines the subroutines just described.

```
algorithm PREFLOW-PUSH;
begin
        PREPROCESS;
        while the network contains an active node do
        begin
                select an active node i;
                PUSH/RELABEL(i);
        end;
end;
```

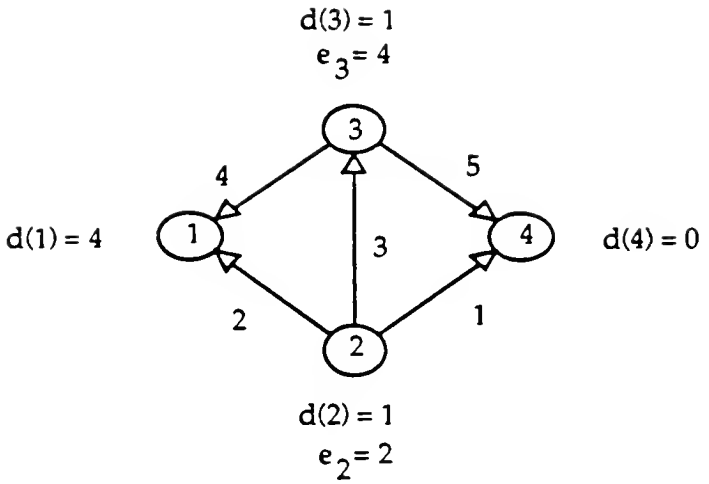
It might be instructive to visualize the generic preflow-push algorithm in terms of a physical network: arcs represent flexible water pipes, nodes represent joints, and the distance function measures how far nodes are above the ground; and in this network, we wish to send water from the source to the sink. In addition, we visualize flow in an

admissible arc as water flowing downhill. Initially, we move the source node upward, and water flows to its neighbors. In general, water flows downhill towards the sink; however, occasionally flow becomes trapped locally at a node that has no downhill neighbors. At this point, we move the node upward, and again water flows downhill towards the sink. Eventually, no flow than can reach the sink. As we continue to move nodes upwards, the remaining excess flow eventually flows back towards the source. The algorithm terminates when all the water flows either into the sink or into the source.

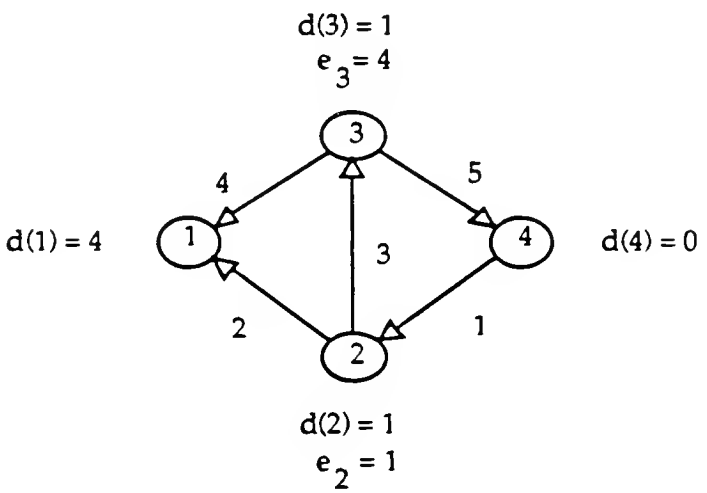
Figure 4.3 illustrates the push/relabel steps applied to the example given in Figure 4.1(a). Figure 4.3(a) specifies the preflow determined by the preprocess step. Suppose the select step examines node 2. Since arc (2, 4) has residual capacity $r_{24} = 1$ and $d(2) = d(4) + 1$, the algorithm performs a (saturating) push of value $\delta = \min\{2, 1\}$ units. The push reduces the excess of node 2 to 1. Arc (2, 4) is deleted from the residual network and arc (4, 2) is added to the residual network. Since node 2 is still an active node, it can be selected again for further pushes. The arc (2, 3) and (2, 1) have positive residual capacities, but they do not satisfy the distance condition. Hence, the algorithm performs a relabel operation and gives node 2 a new distance $d'(2) = \min\{d(3) + 1, d(1) + 1\} = \min\{2, 5\} = 2$.

The preprocessing step accomplishes several important tasks. First, it gives each node adjacent to node s a positive excess, so that the algorithm can begin by selecting some node with positive excess. Second, since the preprocessing step saturates all arcs incident to node s, none of these arcs is admissible and setting $d(s) = n$ will satisfy the validity condition C4.2. Third, since $d(s) = n$ is a lower bound on the length of any shortest path from s to t, the residual network contains no path from s to t. Since distances in d are nondecreasing, we are also guaranteed that in subsequent iterations the residual network will never contain a directed path from s to t, and so there never will be any need to push flow from s again.

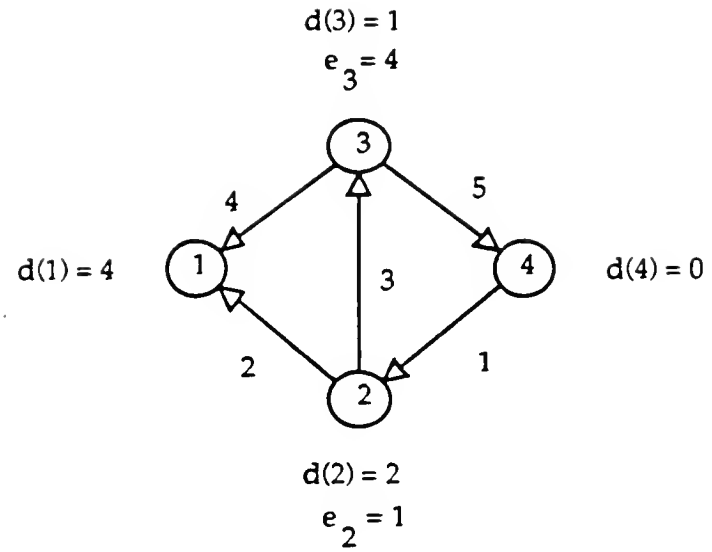
In the push/relabel(i) step, we identify an admissible arc in A(i) using the same data structure we used in the shortest augmenting path algorithm. We maintain with each node i a *current arc* (i, j) which is the current candidate for the push operation. We choose the current arc by sequentially scanning the arc list. We have seen earlier that scanning the arc lists takes O(nm) total time, if the algorithm relabels each node O(n) times.



(a) The residual network after the preprocessing step.



(b) After the execution of step PUSH(2).



(c) After the execution of step RELABEL(2).

Figure 4.3 An illustration of push and relabel steps.

Assuming that the generic preflow-push algorithm terminates, we can easily show that it finds a maximum flow. The algorithm terminates when the excess resides either at the source or at the sink implying that the current preflow is a flow. Since d(s) = n, the residual network contains no path from the source to the sink. This condition is the termination criterion of the augmenting path algorithm, and thus the total flow on arcs directed into the sink is the maximum flow value.

Complexity of the Algorithm

We now analyze the complexity of the algorithm. We begin by establishing one important result: that distance labels are always valid and do not increase too many times. The first of these conclusions follows from Lemma 4.1, because as in the shortest augmenting path algorithm, the preflow-push algorithm pushes flow only on admissible arcs and relabels a node only when no admissible arc emanates from it. The second conclusion follows from the following lemma.

Lemma 4.3. *At any stage of the preflow-push algorithm, each node i with positive excess is connected to node s by a directed path from i to s in the residual network.*

Proof. By the flow decomposition theory, any preflow x can be decomposed with respect to the original network G into nonnegative flows along (i) paths from the source s to t, (ii) paths from s to active nodes, and (iii) the flows around directed cycles. Let i be an

active node relative to the preflow x in G. Then there must be a path P from s to i in the flow decomposition of x, since paths from s to t and flows around cycles do not contribute to the excess at node i. Then the residual network contains the reversal of P (P with the orientation of each arc reversed), and hence a directed path from i to s. ■

This lemma imples that during a relabel step, the algorithm does not minimize over an empty set.

Lemma 4.4. For each node $i \in N$, $d(i) < 2n$.

Proof. The last time the algorithm relabeled node i, it had a positive excess, and hence the residual network contained a path of length at most n–1 from node i to node s. The fact that d(s) = n and condition C4.2 imply that $d(i) \leq d(s) + n - 1 < 2n$. ■

Lemma 4.5. (a) Each distance label increases at most 2n times. Consequently, the total number of relabel steps is at most $2n^2$. (b) The number of saturating pushes is at most nm.

Proof. The proof is very much similar to that of Lemma 4.2. ■

Lemma 4.6. The number of nonsaturating pushes is $O(n^2m)$.

Proof. We prove the lemma using an argument based on potential functions. Let I denote the set of active nodes. Consider the potential function $F = \sum_{i \in I} d(i)$. Since $|I| \leq n$, and $d(i) \leq 2n$ for all $i \in I$, the initial value of F (after the preprocessing step) is at most $2n^2$. At termination, F is zero. During the push/relabel(i) step, one of the following two cases must apply:

Case 1. The algorithm is unable to find an admissible arc along which it can push flow. In this case the distance label of node i increases by $\varepsilon \geq 1$ units. This operation increases F by at most ε units. Since the total increase in d(i) throughout the running time of the algorithm for each node i is bounded by 2n, the total increase in F due to increases in distance labels is bounded by $2n^2$.

Case 2. The algorithm is able to identify an arc on which it can push flow, and so it performs a saturating or a nonsaturating push. A saturating push on arc (i, j) might create a new excess at node j, thereby increasing the number of active nodes by 1, and increasing F by d(j), which may be as much as 2n per saturating push, and hence $2n^2m$ over all saturating pushes. Next note that a nonsaturating push on arc (i, j) does not

increase III. The nonsaturating push will decrease F by d(i) since i becomes inactive, but it simultaneously increases F by d(j) = d(i) - 1 if the push causes node j to become active. If node j was active before the push, then F decreases by an amount d(i). The net decrease in F is at least 1 unit per nonsaturating push.

We summarize these facts. The initial value of F is at most $2n^2$ and the maximum possible increase in F is $2n^2 + 2n^2m$. Each nonsaturating push decreases F by one unit and F always remains nonnegative. Hence, the nonsaturating pushes can occur at most $2n^2 + 2n^2 + 2n^2m = O(n^2m)$ times, proving the lemma. ■

Finally, we indicate how the algorithm keeps track of active nodes for the push/relabel steps. The algorithm maintains a set S of active nodes. It adds to S nodes that become active following a push and are not already in S, and deletes from S nodes that become inactive following a nonsaturating push. Several data structures (for example, doubly linked lists) are available for storing S so that the algorithm can add, delete, or select elements from it in O(1) time. Consequently, it is easy to implement the preflow-push algorithm in $O(n^2m)$ time. We have thus established the following theorem:

Theorem 4.4 The generic preflow-push algorithm runs in $O(n^2m)$ time. ■

A Specialization of the Generic Algorithm

The running time of the generic preflow-push algorithm is comparable to the bound of the shortest augmenting path algorithm. However, the preflow-push algorithm has several nice features, in particular, its flexibility and its potential for further improvements. By specifying different rules for selecting nodes for push/relabel operations, we can derive many different algorithms from the generic version. For example, suppose that we always select an active node with the highest distance label for push/relabel step. Let $h^* = \max \{d(i) : e(i) > 0, i \in N\}$ at some point of the algorithm. Then nodes with distance h^* push flow to nodes with distance h^*-1, and these nodes, in turn, push flow to nodes with distance h^*-2, and so on. If a node is relabeled then excess moves up and then gradually comes down. Note that if the algorithm relabels no node during n consecutive node examinations, then all excess reaches the sink node and the algorithm terminates. Since the algorithm requires $O(n^2)$ relabel operations, we immediately obtain a bound of $O(n^3)$ on the number of node examinations. Each node examination entails at most one nonsaturating push. Consequently, this algorithm performs $O(n^3)$ nonsaturating pushes.

variable *level* which is an upper bound on the highest index r for which LIST(r) is nonempty. We can store these lists as doubly linked lists so that adding, deleting, or selecting an element takes O(1) time. We identify the highest indexed nonempty list starting at LIST(level) and sequentially scanning the lower indexed lists. We leave it as an exercise to show that the overall effort needed to scan the lists is bounded by n plus the total increase in the distance labels which is $O(n^2)$. The following theorem is now evident.

Theorem 4.5 The preflow-push algorithm that always pushes flow from an active node with the highest distance label runs in $O(n^3)$ time. ■

The $O(n^3)$ bound for the highest label preflow push algorithm is straightforward, and can be improved. Researchers have shown using more clever analysis that the highest label preflow push algorithm in fact runs in $O(n^2 \sqrt{m})$ time. We will next describe another implementation of the generic preflow-push algorithm that dramatically reduces the number of nonsaturating pushes, from $O(n^2m)$ to $O(n^2 \log U)$. Recall that U represents the largest arc capacity in the network. We refer to this algorithm as the *excess-scaling algorithm* since it is based on scaling the node excesses.

4.5 Excess-Scaling Algorithm

The generic preflow-push algorithm allows flows at each intermediate step to violate mass balance equations. By pushing flows from active nodes, the algorithm attempts to satisfy the mass balance equations. The function $e_{max} = \max \{e(i) : i \text{ is an active node}\}$ is one measure of the infeasibility of a preflow. Note, though, that during the execution of the generic algorithm, we would observe no particular pattern in e_{max}, except that e_{max} eventually decreases to value 0. In this section, we develop an *excess-scaling technique* that systematically reduces e_{max} to 0.

The excess-scaling algorithm is based on the following ideas. Let Δ denote an upper bound on e_{max} we refer to this bound as the *excess-dominator*. The excess-scaling algorithm pushes flow from nodes whose excess is $\Delta/2 \geq e_{max}/2$. This choice assures that during nonsaturating pushes the algorithm sends relatively large excess closer to the sink. Pushes carrying small amounts of flow are of little benefit and can cause bottlenecks that retards the algorithm's progress.

The algorithm also does not allow the maximum excess to increase beyond Δ. This algorithmic strategy may prove to be useful for the following reason. Suppose

The algorithm also does not allow the maximum excess to increase beyond Δ. This algorithmic strategy may prove to be useful for the following reason. Suppose several nodes send flow to a single node j, creating a very large excess. It is likely that node j could not send the accumulated flow closer to the sink, and thus the algorithm will need to increase its distance and return much of its excess back toward the source. Thus, pushing too much flow to any node is likely to be a wasted effort.

The excess-scaling algorithm has the following algorithmic description.

```
algorithm EXCESS-SCALING;
begin
    PREPROCESS;
    K := 2^⌈log U⌉;
    for k := K down to 0 do
    begin (Δ-scaling phase)
        Δ := 2^k;
        while the network contains a node i with e(i) > Δ/2 do
            perform push/relabel(i) while ensuring that no node excess
                exceeds Δ;
    end;
end;
```

The algorithm performs a number of scaling phases with the value of the excess-dominator Δ decreasing from phase to phase. We refer to a specific scaling phase with a certain value of Δ as the *Δ-scaling phase*. Initially, $\Delta = 2^{\lceil \log U \rceil}$ when the logarithm has base 2. Thus, $U \le \Delta \le 2U$. During the Δ-scaling phase, $\Delta/2 < e_{max} \le \Delta$ and e_{max} may vary up and down during the phase. When $e_{max} \le \Delta/2$, a new scaling phase begins. After the algorithm has peformed $\lceil \log U \rceil + 1$ scaling phases, e_{max} decreases to value 0 and we obtain the maximum flow.

The excess-scaling algorithm uses the same step push/relabel(i) as in the generic preflow-push algorithm, but with one slight difference: instead of pushing $\delta = \min \{e(i), r_{ij}\}$ units of flow, it pushes $\delta = \min \{e(i), r_{ij}, \Delta - e(j)\}$ units. This change will ensure that the algorithm permits no excess to exceed Δ. The algorithm uses the following node selection rule to guarantee that no node excess exceeds Δ.

Selection Rule. Among all nodes with excess of more than Δ/2, select a node with minimum distance label (breaking ties arbitrarily).

Lemma 4.7. The algorithm satisfies the following two conditions:

C4.3. Each nonsaturating push sends at least $\Delta/2$ units of flow.
C4.4. No excess ever exceeds Δ.

Proof. For every push on arc (i, j), we have $e(i) > \Delta/2$ and $e(j) \le \Delta/2$, since node i is a node with smallest distance label among nodes whose excess is more than $\Delta/2$, and $d(j) = d(i) - 1 < d(i)$ since arc (i, j) is admissible. Hence, by sending $\min \{e(i), r_{ij}, \Delta - e(j)\} \ge \min \{\Delta/2, r_{ij}\}$ units of flow, we ensure that in a nonsaturating push the algorithm sends at least $\Delta/2$ units of flow. Further, the push operation increases only e(j). Let e'(j) be the excess at node j after the push. Then $e'(j) = e(j) + \min \{e(i), r_{ij}, \Delta - e(j)\} \le e(j) + \Delta - e(j) \le \Delta$. All node excesses thus remain less than or equal to Δ. ■

Lemma 4.8. The excess-scaling algorithm performs $O(n^2)$ nonsaturating pushes per scaling phase and $O(n^2 \log U)$ pushes in total.

Proof. Consider the potential function $F = \sum_{i \in N} e(i)\, d(i)/\Delta$. Using this potential function we will establish the first assertion of the lemma. Since the algorithm has O(log U) scaling phases, the second assertion is a consequence of the first. The initial value of F at the beginning of the Δ-scaling phase is bounded by $2n^2$ because e(i) is bounded by Δ and d(i) is bounded by 2n. During the push/relabel(i) step, one of the following two cases must apply:

Case 1. The algorithm is unable to find an admissible arc along which it can push flow. In this case the distance label of node i increases by $\varepsilon \ge 1$ units. This relabeling operation increases F by at most ε units because $e(i) \le \Delta$. Since for each i, the total increase in d(i) throughout the running of the algorithm is bounded by 2n (by Lemma 4.4), the total increase in F due to the relabeling of nodes is bounded by $2n^2$ in the Δ-scaling phase (actually, the increase in F due to node relabelings is at most $2n^2$ over *all* scaling phases).

Case 2. The algorithm is able to identify an arc on which it can push flow and so it performs either a saturating or a nonsaturating push. In either case, F decreases. A nonsaturating push on arc (i, j) sends at least $\Delta/2$ units of flow from node i to node j and since d(j) = d(i) - 1, after this operation F decreaases by at least 1/2 units. Since the initial value of F at the beginning of a Δ-scaling phase is at most $2n^2$ and the increases in F during this scaling phase sum to at most $2n^2$ (from Case 1), the number of nonsaturating pushes is bounded by $8n^2$. ■

This lemma implies a bound of $O(nm + n^2 \log U)$ for the excess-scaling algorithm since we have already seen that all other operations -- such as saturating pushes, relabel operations and finding admissible arcs -- require O(nm) time. Up to this point, we have ignored the method needed to identify a node with the minimum distance label among nodes with excess more than $\Delta/2$. Making this identification is easy, if we use a scheme similar to the one used in the preflow-push method in Section 4.4 to find a node with the highest distance label. We maintain the lists LIST(r) = {i ∈ N : e(i) > $\Delta/2$ and d(i) = r}, and a variable *level* which is a lower bound on the smallest index r for which LIST(r) is nonempty. We identify the lowest indexed nonempty list starting at LIST(level) and sequentially scanning the higher indexed lists. We leave as an exercise to show that the overall effort needed to scan the lists is bounded by the number of pushes performed by the algorithm plus O(n log U) and, hence, is not a bottleneck operation. With this observation, we can summarize our discussion by the following result.

Theorem 4.6 The preflow-push algorithm with excess-scaling runs in $O(nm + n^2 \log U)$ time. ■

Networks with Lower Bounds on Flows

To conclude this section, we show how to solve maximum flow problems with nonnegative lower bounds on flows. Let $l_{ij} \geq 0$ denote the lower bound for flow on any arc (i, j) ∈ A. Although the maximum flow problem with zero lower bounds always has a feasible solution, the problem with nonnegative lower bounds could be infeasible. We can, however, determine the feasiblility of this problem by solving a maximum flow problem with zero lower bounds as follows. We set $x_{ij} = l_{ij}$ for each arc (i, j) ∈ A. This choice gives us a pseudoflow with e(i) representing the excess or deficit of any node i ∈ N. (We refer the reader to Section 5.4 for the definition of a pseudoflow with both excesses and deficits). We introduce a *super source*, node s*, and a *super sink*, node t*. For each node i with e(i) > 0, we add an arc (s*,i) with capacity e(i), and for each node i with e(i) < 0, we add an arc (i, t*) with capacity –e(i). We then solve a maximum flow problem from s* to t*. Let x^* denote the maximum flow and v^* denote the maximum flow value in the transformed network. If $v^* = \sum_{\{i:\, e(i) > 0\}} e(i)$, then the original problem is feasible and choosing the flow on each arc (i, j) as $x^*_{ij} + l_{ij}$ is a feasible flow; otherwise, the problem is infeasible.

Once we have found a feasible flow, we apply any of the maximum flow algorithms with only one change: initially define the residual capacity of an arc (i, j) as $r_{ij} = (u_{ij} - x_{ij}) + (x_{ji} - l_{ji})$. The first and second terms in this expression denote, respectively, the residual capacity for increasing flow on arc (i, j) and for decreasing flow on arc (j, i). It is possible to establish the optimality of the solution generated by the algorithm by generalizing the max-flow min-cut theorem to accomodate situations with lower bounds. These observations show that it is possible to solve the maximum flow problem with nonnegative lower bounds by two applications of the maximum flow algorithms we have already discussed.

5. MINIMUM COST FLOWS

In this section, we consider algorithmic approaches for the minimum cost flow problem. We consider the following node-arc formulation of the problem.

$$\text{Minimize} \sum_{(i,j) \in A} c_{ij} x_{ij} \tag{5.1a}$$

subject to

$$\sum_{\{j : (i,j) \in A\}} x_{ij} - \sum_{\{j : (j,i) \in A\}} x_{ji} = b(i), \text{ for all } i \in N, \tag{5.1b}$$

$$0 \le x_{ij} \le u_{ij}, \text{ for each } (i,j) \in A. \tag{5.1c}$$

We assume that the lower bounds l_{ij} on arc flows are all zero and that arc costs are nonnegative. Let $C = \max \{ c_{ij} : (i,j) \in A \}$ and $U = \max [\max \{ |b(i)| : i \in N \}, \max \{ u_{ij} : (i,j) \in A \}]$. The transformations T1 and T3 in Section 2.4 imply that these assumptions do not impose any loss of generality. We remind the reader of our blanket assumption that all data (cost, supply/demand and capacity) are integral. We also assume that the minimum cost flow problem satisfies the following two conditions.

A5.1. Feasibility Assumption. We assume that $\sum_{i \in N} b(i) = 0$ *and that the minimum cost flow problem has a feasible solution.*

We can ascertain the feasibility of the minimum cost flow problem by solving a maximum flow problem as follows. Introduce a *super source* node s^*, and a *super sink* node t^*. For each node i with $b(i) > 0$, add an arc (s^*, i) with capacity b(i), and for each node i with $b(i) < 0$, add an arc (i, t^*) with capacity $-b(i)$. Now solve a maximum flow problem from s^* to t^*. If the maximum flow value equals $\sum_{\{i : b(i) > 0\}} b(i)$ then the minimum cost flow problem is feasible; otherwise, it is infeasible.

A5.2. Connectedness Assumption. We assume that the network G contains an uncapacitated directed path (i.e., each arc in the path has infinite capacity) between every pair of nodes.

We impose this condition, if necessary, by adding *artificial* arcs (1, j) and (j, 1) for each $j \in N$ and assigning a large cost and a very large capacity to each of these

arcs. No such arc would appear in a minimum cost solution unless the problem contains no feasible solution without artificial arcs.

Our algorithms rely on the concept of residual networks. The residual network G(x) corresponding to a flow x is defined as follows: We replace each arc (i, j) $\in$ A by two arcs (i, j) and (j, i). The arc (i, j) has cost c_{ij} and *residual capacity* $r_{ij} = u_{ij} - x_{ij}$, and the arc (j, i) has cost $-c_{ij}$ and residual capacity $r_{ji} = x_{ij}$. The residual network consists *only* of arcs with positive residual capacity.

The concept of residual networks poses some notational difficulties. For example, if the original network contains both the arcs (i, j) and (j, i), then the residual network may contain two arcs from node i to node j and/or two arcs from node j to node i with possibly different costs. Our notation for arcs assumes that at most one arc joins one node to any other node. By using more complex notation, we can easily treat this more general case. However, rather than changing our notation, we will assume that parallel arcs never arise (or, by inserting extra nodes on parallel arcs, we can produce a network without any parallel arcs).

Observe that any directed cycle in the residual network G(x) is an augmenting cycle with respect to the flow x and vice-versa (see Section 2.1 for the definition of augmenting cycle). This equivalence implies the following alternate statement of Theorem 2.4.

Theorem 5.1. *A feasible flow x is an optimum flow if and only if the residual network G(x) contains no negative cost directed cycle.* ■

5.1. Duality and Optimality Conditions

As we have seen in Section 1.2, due to its special structure, the minimum cost flow problem has a number of important theoretical properties. The linear programming dual of this problem inherits many of these properties. Moreover, the minimum cost flow problem and its dual have, from a linear programming point of view, rather simple complementary slackness conditions. In this section, we formally state the linear programming dual problem and derive the complementary slackness conditions.

We consider the minimum cost flow problem (5.1) assuming that $u_{ij} > 0$ for each arc $(i, j) \in A$. It is possible to show that this assumption imposes no loss of generality. We associate a dual variable $\pi(i)$ with the mass balance constraint of node i in (5.1b). Since one of the constraints in (5.1b) is redundant, we can set one of these dual variables to an arbitrary value. We, therefore , assume that $\pi(1) = 0$. Further, we associate a dual variable δ_{ij} with the upper bound constraint of arc (i, j) in (5.1c). The dual problem to (5.1) is:

$$\text{Maximize} \sum_{i \in N} b(i)\,\pi(i) - \sum_{(i,j) \in A} u_{ij}\,\delta_{ij} \tag{5.2a}$$

subject to

$$\pi(i) - \pi(j) - \delta_{ij} \le c_{ij}, \text{ for all } (i, j) \in A, \tag{5.2b}$$

$$\delta_{ij} \ge 0, \text{ for all } (i, j) \in A, \tag{5.2c}$$

and $\pi(i)$ are unrestricted.

The complementary slackness conditions for this primal-dual pair are:

$$x_{ij} > 0 \Rightarrow \pi(i) - \pi(j) - \delta_{ij} = c_{ij}, \tag{5.3}$$

$$\delta_{ij} > 0 \Rightarrow x_{ij} = u_{ij}. \tag{5.4}$$

These conditions are equivalent to the following optimality conditions:

$$x_{ij} = 0 \Rightarrow \pi(i) - \pi(j) \le c_{ij}, \tag{5.5}$$

$$0 < x_{ij} < u_{ij} \Rightarrow \pi(i) - \pi(j) = c_{ij}, \tag{5.6}$$

$$x_{ij} = u_{ij} \Rightarrow \pi(i) - \pi(j) \ge c_{ij}. \tag{5.7}$$

To see this equivalence, suppose that $0 < x_{ij} < u_{ij}$ for some arc (i, j). The condition (5.3) implies that

$$\pi(i) - \pi(j) - \delta_{ij} = c_{ij}. \tag{5.8}$$

Since $x_{ij} < u_{ij}$, (5.4) implies that $\delta_{ij} = 0$; substituting this result in (5.8) yields (5.6). Whenever $x_{ij} = u_{ij} > 0$ for some arc (i, j), (5.3) implies that $\pi(i) - \pi(j) - \delta_{ij} = c_{ij}$.

Substituting $\delta_{ij} \geq 0$ in this equation gives (5.7). Finally, if $x_{ij} = 0 < u_{ij}$ for some arc (i, j) then (5.4) imples that $\delta_{ij} = 0$ and substituting this result in (5.2b) gives (5.5).

We define the *reduced cost* of an arc (i, j) as $\bar{c}_{ij} = c_{ij} - \pi(i) + \pi(j)$. The conditions (5.5) – (5.7) imply that a pair x, π of flows and node potentials is optimal if it satisfies the following conditions:

C5.1 x is feasible.
C5.2 If $\bar{c}_{ij} > 0$, then $x_{ij} = 0$.
C5.3 If $\bar{c}_{ij} = 0$, then $0 \leq x_{ij} \leq u_{ij}$.
C5.4 If $\bar{c}_{ij} < 0$, then $x_{ij} = u_{ij}$.

Observe that the condition C5.3 follows from the conditions C5.1, C5.2 and C5.4; however, we retain it for the sake of completeness. These conditions, when stated in terms of the residual network, simplify to:

C5.5 (Primal feasibility) x is feasible.
C5.6 (Dual feasibility) $\bar{c}_{ij} \geq 0$ for each arc (i, j) in the residual network G(x).

Note that the condition C5.6 subsumes C5.2, C5.3, and C5.4. To see this result, note that if $\bar{c}_{ij} > 0$ and $x_{ij} > 0$ for some arc (i, j) in the original network, then the residual network would contain arc (j, i) with $\bar{c}_{ji} = -\bar{c}_{ij}$. But then $\bar{c}_{ji} < 0$, contradicting C5.6. A similar contradiction arises if $\bar{c}_{ij} < 0$ and $x_{ij} < u_{ij}$ for some (i, j) in A.

It is easy to establish the equivalence between these optimality conditions and the condition stated in Theorem 5.1. Consider any pair x, π of flows and node potentials satisfying C5.5 and C5.6. Let W be any directed cycle in the residual network. Condition C5.6 implies that $\sum_{(i,j) \in W} \bar{c}_{ij} \geq 0$. Further, $0 \leq \sum_{(i,j) \in W} \bar{c}_{ij} = \sum_{(i,j) \in W} c_{ij} + \sum_{(i,j) \in W} (-\pi(i) + \pi(j))$

$= \sum_{(i,j) \in W} c_{ij}$. Hence, the residual network contains no negative cost cycle.

To see the converse, suppose that x is feasible and G(x) does not contain a negative cycle. Then in the residual network the shortest distances from node 1, with respect to the arc lengths c_{ij}, are well defined. Let d(i) denote the shortest distance from node 1 to node i. The shortest path optimality condition C3.2 implies that $d(j) \leq d(i) + c_{ij}$

for all (i, j) in G(x). Let $\pi = -d$. Then $0 \le c_{ij} + d(i) - d(j) = c_{ij} - \pi(i) + \pi(j) = \bar{c}_{ij}$ for all (i, j) in G(x). Hence, the pair x, π satisfies C5.5 and C5.6.

5.2. Relationship to Shortest Path and Maximum Flow Problems

The minimum cost flow problem generalizes both the shortest path and maximum flow problems. The shortest path problem from node s to all other nodes can be formulated as a minimum cost flow problem by setting $b(1) = (n - 1)$, $b(i) = -1$ for all $i \neq s$, and $u_{ij} = \infty$ for each $(i, j) \in A$ (in fact, setting u_{ij} equal to any integer greater than $(n - 1)$ will suffice if we wish to maintain finite capacities). Similarly, the maximum flow problem from node s to node t can be transformed to the minimum cost flow problem by introducing an additional arc (t, s) with $c_{ts} = -1$ and $u_{ts} = \infty$ (in fact, $u_{ts} = m \cdot \max \{u_{ij} : (i, j) \in A\}$ would suffice), and setting $c_{ij} = 0$ for each arc $(i, j) \in A$. Thus, algorithms for the minimum cost flow problem solve both the shortest path and maximum flow problems as special cases.

Conversely, algorithms for the shortest path and maximum flow problems are of great use in solving the minimum cost flow problem. Indeed, many of the algorithms for the minimum cost flow problem either explicitly or implicitly use shortest path and/or maximum flow algorithms as subroutines. Consequently, improved algorithms for these two problems have led to improved algorithms for the minimum cost flow problem. This relationship will be more transparent when we discuss algorithms for the minimum cost flow problem. We have already shown in Section 5.1 how to obtain an optimum dual solution from an optimum primal solution by solving a single shortest path problem. We now show how to obtain an optimal primal solution from an optimal dual solution by solving a single maximum flow problem.

Suppose that π is an optimal dual solution and $\bar{c}$ is the vector of reduced costs. We define the *cost-residual network* $G^* = (N, A^*)$ as follows. The nodes in G^* have the same supply/demand as the nodes in G. Any arc $(i, j) \in A^*$ has an upper bound u_{ij}^* as well as a lower bound l_{ij}^*, defined as follows:

(i) For each (i, j) in A with $\bar{c}_{ij} > 0$, A* contains an arc (i, j) with $u_{ij}^* = l_{ij}^* = 0$.

(ii) For each (i, j) in A with $\bar{c}_{ij} < 0$, A* contains an arc (i, j) with $u_{ij}^* = l_{ij}^* = u_{ij}$.

(iii) For each (i, j) in A with $\bar{c}_{ij} = 0$, A* contains an arc (i, j) with $u_{ij}^* = u_{ij}$ and $l_{ij}^* = 0$.

The lower and upper bounds on arcs in the cost-residual network G* are defined so that any flow in G* satisfies the optimality conditions C5.2-C5.4. If $\bar{c}_{ij} > 0$ for some (i, j) $\in$ A, then condition C5.2 dictates that $x_{ij} = 0$ in the optimum flow. Similarly, if $\bar{c}_{ij} < 0$ for some (i, j) $\in$ A, then C5.4 implies the flow on arc (i, j) must be at the arc's upper bound in the optimum flow. If $\bar{c}_{ij} = 0$, then any flow value will satisfy the condition C5.3.

Now the problem is reduced to finding a feasible flow in the cost-residual network that satisfies the lower and upper bound restrictions of arcs and, at the same time, meets the supply/demand constraints of the nodes. We first eliminate the lower bounds of arcs as described in Section 2.4 and then transform this problem to a maximum flow problem as described in assumption A5.1. Let x^* denote the maximum flow in the transformed network. Then x^*+l^* is an optimum solution of the minimum cost problem in G.

5.3. Negative Cycle Algorithm

Operations researchers, computer scientists, electrical engineers and many others have extensively studied the minimum cost flow problem and have proposed a number of different algorithms to solve this problem. Notable examples are the negative cycle, successive shortest path, primal-dual, out-of-kilter, primal simplex and scaling-based algorithms. In this and the following sections, we discuss most of these important algorithms for the minimum cost flow problem and point out relationships between them. We first consider the negative cycle algorithm.

The negative cycle algorithm maintains a primal feasible solution x and strives to attain dual feasibility. It does so by identifying negative cost directed cycles in the residual network G(x) and augmenting flows in these cycles. The algorithm terminates when the residual network contains no negative cost cycles. Theorem 5.1 implies that when the algorithm terminates, it has found a minimum cost flow.

```
algorithm NEGATIVE CYCLE;
begin
    establish a feasible flow x in the network;
    while G(x) contains a negative cycle do
    begin
        use some algorithm to identify a negative cycle W;
        δ := min {r_ij : (i, j) ∈ W};
        augment δ units of flow along the cycle W and update G(x);
    end;
end;
```

A feasible flow in the network can be found by solving a maximum flow problem as explained just after assumption A5.1. One algorithm for identifying a negative cost cycle is the label correcting algorithm for the shortest path problem, described in Section 3.4, which requires O(nm) time to identify a negative cycle. Every iteration reduces the flow cost by at least one unit. Since mCU is an upper bound on an initial flow cost and zero is a lower bound on the optimum flow cost, the algorithm terminates after at most O(mCU) iterations and requires $O(nm^2CU)$ time in total.

This algorithm can be improved in the following three ways (which we briefly summarize).

(i) Identifying a negative cost cycle in effort much less than O(nm) time. The simplex algorithm (to be discussed later) nearly achieves this objective. It maintains a tree solution and node potentials that enable it to identify a negative cost cycle in O(m) effort. However, due to degeneracy, the simplex algorithm cannot necessarily send a positive amount of flow along this cycle.

(ii) Identifying a negative cost cycle with maximum improvement in the objective function value. The improvement in the objective function due to the augmentation along a cycle W is $\left(-\sum_{(i,j) \in W} c_{ij}\right)\left(\min \{r_{ij} : (i, j) \in W\}\right)$. Let x be some flow and x^* be an optimum flow. The augmenting cycle theorem (Theorem 2.3) implies that x^* equals x plus the flow on at most m augmenting cycles with respect to x. Further, improvements in cost due to flow augmentations on these augmenting cycles sum to $cx - cx^*$. Consequently, at least one augmenting cycle with respect to x must decrease the objective function by at least $(cx - cx^*)/m$. Hence, if the algorithm always augments flow along a

cycle with maximum improvement, then Lemma 1.1 implies that the method would obtain an optimum flow within O(m log mCU) iterations. Finding a maximum improvement cycle is a difficult problem, but a modest variation of this approach yields a polynomial time algorithm for the minimum cost flow problem.

(iii) Identifying a negative cost cycle with minimum mean cost. We define the *mean cost* of a cycle as its cost divided by the number of arcs it contains. A *minimum mean cycle* is a cycle whose mean cost is as small as possible. It is possible to identify a minimum mean cycle in O(nm) or O($\sqrt{n}$ m log nC) time. Recently, researchers have shown that if the negative cycle algorithm always augments the flow along a minimum mean cycle, then from one iteration to the next, the minimum mean cycle value is nondecreasing; moreover, its absolute value decreases by a factor of 1-(1/n) within m iterations. Since the mean cost of the minimum mean (negative) cycle is bounded from below by -C and bounded from above by -1/n, Lemma 1.1 implies that this algorithm will terminate in O(nm log nC) iterations.

5.4. Successive Shortest Path Algorithm

The negative cycle algorithm maintains primal feasibility of the solution at every step and attempts to achieve dual feasibility. In contrast, the successive shortest path algorithm maintains dual feasibility of the solution at every step and strives to attain primal feasibility. It maintains a solution x that satisfies the nonnegativity and capacity constraints, but violates the supply/demand constraints of the nodes. At each step, the algorithm selects a node i with extra supply and a node j with unfulfilled demand and sends flow from i to j along a shortest path in the residual network. The algorithm terminates when the current solution satisfies all the supply/demand constraints.

A *pseudoflow* is a function $x : A \rightarrow R$ satisfying only the capacity and nonnegativity constraints. For any pseudoflow x, we define the *imbalance* of node i as

$$e(i) = b(i) + \sum_{\{j:\,(j,i)\in A\}} x_{ji} - \sum_{\{j:\,(i,j)\in A\}} x_{ij}, \text{ for all } i \in N.$$

If e(i) > 0 for some node i, then e(i) is called the *excess* of node i, if e(i) < 0, then -e(i) is called the *deficit*. A node i with e(i) = 0 is called *balanced*. Let S and T denote the

sets of excess and deficit nodes respectively. The residual network corresponding to a pseudoflow is defined in the same way that we define the residual network for a flow.

The successive shortest path algorithm successively augments flow along shortest paths computed with respect to the reduced costs $\bar{c}_{ij}$. Observe that for any directed path P from a node k to a node l, $\sum_{(i,j) \in P} \bar{c}_{ij} = \sum_{(i,j) \in P} c_{ij} - \pi(l) + \pi(k)$. Hence, the node potentials change all path lengths between a specific pair of nodes by a constant amount, and the shortest path with respect to c_{ij} is the same as the shortest path with respect to $\bar{c}_{ij}$. The correctness of the successive shortest path algorithm rests on the following result.

Lemma 5.1. Suppose a pseudoflow x satisfies the dual feasibility condition C5.6 with respect to the node potentials π. Furthermore, suppose that x' is obtained from x by sending flow along a shortest path from a node k to a node l in G(x). Then x' also satisfies the dual feasibility conditions with respect to some node potentials.

Proof. Since x satisfies the dual feasibility conditions with respect to the node potentials π, we have $\bar{c}_{ij} \geq 0$ for all (i, j) in G(x). Let d(v) denote the shortest path distances from node k to any node v in G(x) with respect to the arc lengths $\bar{c}_{ij}$. We claim that x also satisfies the dual feasibility conditions with respect to the potentials $\pi' = \pi - d$. The shortest path optimality conditions (i.e., C3.2) imply that

$$d(j) \leq d(i) + \bar{c}_{ij}, \text{ for all } (i, j) \text{ in } G(x).$$

Substituting $\bar{c}_{ij} = c_{ij} - \pi(i) + \pi(j)$ in these conditions and using $\pi'(i) = \pi(i) - d(i)$ yields

$$\bar{c}_{ij}' = c_{ij} - \pi'(i) + \pi'(j) \geq 0, \text{ for all } (i, j) \text{ in } G(x).$$

Hence, x satisfies C5.6 with respect to the node potentials π'. Next note that $\bar{c}_{ij}' = 0$ for every arc (i, j) on the shortest path P from node k to node l, since $d(j) = d(i) + \bar{c}_{ij}$ for every arc $(i, j) \in P$ and $\bar{c}_{ij} = c_{ij} - \pi(i) + \pi(j)$.

We are now in a position to prove the lemma. Augmenting flow along any arc in P maintains the dual feasibility condition C5.6 for this arc. Augmenting flow on an arc (i, j) may add its reversal (j, i) to the residual network. But since $\bar{c}_{ij} = 0$ for each arc $(i, j) \in P$, $\bar{c}_{ji} = 0$, and so arc (j, i) also satisfies C5.6. ■

The node potentials play a very important role in this algorithm. Besides using them to prove the correctness of the algorithm, we use them to ensure that the arc

lengths are nonnegative, thus enabling us to solve the shortest path subproblems more efficiently. The following formal statement of the successive shortest path algorithm summarizes the steps of this method.

```
algorithm SUCCESSIVE SHORTEST PATH;
begin
    x := 0 and π := 0;
    compute imbalances e(i) and initialize the sets S and T;
    while S ≠ ø do
    begin
        select a node k ∈ S and a node l ∈ T;
        determine shortest path distances d(j) from node k to all
          other nodes in G(x) with respect to the reduced costs c̄ij;
        let P denote a shortest path from k to l;
        update π := π – d;
        δ := min [ e(k), -e(l), min { rij : (i, j) ∈ P } ];
        augment δ units of flow along the path P;
        update x, S and T;
    end;
end;
```

To initialize the algorithm, we set x = 0, which is a feasible pseudoflow and satisfies C5.6 with respect to the node potentials $\pi = 0$ since, by assumption, all arc lengths are nonnegative. Also, if $S \neq \emptyset$, then $T \neq \emptyset$ because the sum of excesses always equals the sum of deficits. Further, the connectedness assumption implies that the residual network G(x) contains a directed path from node k to node *l*. Each iteration of this algorithm solves a shortest path problem with nonnegative arc lengths and reduces the supply of some node by at least one unit. Consequently, if U is an upper bound on the largest supply of any node, the algorithm terminates in at most nU iterations. Since the arc lengths $\bar{c}_{ij}$ are nonnegative, the shortest path problem at each iteration can be solved using Dijkstra's algorithm. So the overall complexity of this algorithm is O(nU·S(n, m, C)), where S(n, m, C) is the time taken by Dijkstra's algorithm. Currently, the best strongly polynomial-time bound to implement Dijkstra's algorithm is O(m + n log n) and the best (weakly) polynomial time bound is $O(\min \{m \log \log C, m + n\sqrt{\log C}\})$. The successive shortest path algorithm is pseudopolynomial time since it is polynomial in n, m and the largest supply U. The algorithm is, however, polynomial

time for the assignment problem, a special case of the minimum cost flow problem for which U = 1. In Section 5.7, we will develop a polynomial time algorithm for the minimum cost flow problem using the successive shortest path algorithm in conjunction with scaling.

5.5. Primal-Dual and Out-of-Kilter Algorithms

The *primal-dual algorithm* is very similar to the successive shortest path problem, except that instead of sending flow on only one path during an iteration, it might send flow along many paths. To explain the primal-dual algorithm, we transform the minimum cost flow problem into a single-source and single-sink problem (possibly by adding nodes and arcs as in the assumption A5.1). At every iteration, the primal-dual algorithm solves a shortest path problem from the source to update the node potentials (i.e., as before, each $\pi(j)$ becomes $\pi(j) - d(j)$) and then solves a maximum flow problem to send the maximum possible flow from the source to the sink *using only arcs with zero reduced cost*. The algorithm guarantees that the excess of some node strictly decreases at each iteration, and also assures that the node potential of the sink strictly decreases. The latter observation follows from the fact that after we have solved the maximum flow problem, the network contains no path from the source to the sink in the residual network consisting entirely of arcs with zero reduced costs; consequently, in the next iteration $d(t) \geq 1$. These observations give a bound of min {nU, nC} on the number of iterations since the magnitude of each node potential is bounded by nC. This bound is better than that of the successive shortest path algorithm, but, of course, the algorithm incurs the additional expense of solving a maximum flow problem at each iteration. Thus, the algorithm has an overall complexity of O(min (nU S(n, m, C), nC M(n, m, U)), where S(n, m, C) and M(n, m, U) respectively denote the solution times of shortest path and maximum flow algorithms.

The successive shortest path and primal-dual algorithms maintain a solution that satisfies the dual feasibility conditions and the flow bound constraints, but that violates the mass balance constraints. These algorithms iteratively modify the flow and potentials so that the flow at each step comes closer to satisfying the mass balance constraints. However, we could just as well have violated other constraints at intermediate steps. The *out-of-kilter algorithm* satisfies only the mass balance constraints and may violate the dual feasibility conditions and the flow bound restrictions. The basic idea is to drive the flow on an arc (i, j) to u_{ij} if $\bar{c}_{ij} < 0$, drive the flow to zero if $\bar{c}_{ij} > 0$, and to permit any flow between 0 and u_{ij} if $\bar{c}_{ij} = 0$. The *kilter number*, represented by k_{ij},

k_{ij} of an arc (i, j) is defined as the minimum increase or decrease in the flow necessary to satisfy its flow bound constraint and dual feasibility condition. For example, for an arc (i, j) with $\bar{c}_{ij} > 0$, $k_{ij} = |x_{ij}|$ and for an arc (i, j) with $\bar{c}_{ij} < 0$, $k_{ij} = |u_{ij} - x_{ij}|$. An arc with $k_{ij} = 0$ is said to be *in-kilter*. At each iteration, the out-of-kilter algorithm reduces the kilter number of at least one arc; it terminates when all arcs are in-kilter. Suppose the kilter number of an arc (i, j) would decrease by increasing flow on the arc. Then the algorithm would obtain a shortest path P from node j to node i in the residual network and augment at least one unit of flow in the cycle $P \cup \{(i, j)\}$. The proof of the correctness of this algorithm is similar to, but more detailed than, that of the successive shortest path algorithm.

5.6. Network Simplex Algorithm

The network simplex algorithm for the minimum cost flow problem is a specialization of the bounded variable primal simplex algorithm for linear programming. The special structure of the minimum cost flow problem offers several benefits, particularly, streamlining of the simplex computations and eliminating the need to explicitly maintain the simplex tableau. The tree structure of the basis (see Section 2.3) permits the algorithm to achieve these efficiencies. The advances made in the last two decades for maintaining and updating the tree structure efficiently have substantially improved the speed of the algorithm. Through extensive empirical testing, researchers have also improved the performance of the simplex algorithm by developing various heuristic rules for identifying entering variables. Though no version of the primal network simplex algorithm is known to run in polynomial time, its best implementations are empirically comparable to or better than other minimum cost flow algorithms.

In this section, we describe the network simplex algorithm in detail. We first define the concept of a *basis structure* and describe a data structure to store and to manipulate the basis, which is a spanning tree. We then show how to compute arc flows and node potentials for any basis structure. We next discuss how to perform various simplex operations such as the selection of entering arcs, leaving arcs and pivots using the tree data structure. Finally, we show how to guarantee the finiteness of the network simplex algorithm.

The network simplex algorithm maintains a basic feasible solution at each stage. A basic solution of the minimum cost flow problem is defined by a triple (B, L, U); B, L and U partition the arc set A. The set B denotes the set of *basic arcs*, i.e., arcs of a spanning tree, and L and U respectively denote the sets of *nonbasic arcs* at their lower and upper bounds. We refer to the triple (B, L, U) as a *basis structure*. A basis structure (B, L, U) is called *feasible* if by setting $x_{ij} = 0$ for each $(i, j) \in L$, and setting $x_{ij} = u_{ij}$ for each $(i, j) \in U$, the problem has a feasible solution satisfying (5.1b) and (5.1c). A feasible basis structure (B, L, U) is called an *optimum* basis structure if it is possible to obtain a set of node potentials π so that the reduced costs defined by $\bar{c}_{ij} = c_{ij} - \pi(i) + \pi(j)$ satisfy the following optimality conditions:

$$\bar{c}_{ij} = 0 \text{ , for each } (i, j) \in B, \tag{5.9}$$

$$\bar{c}_{ij} \geq 0 \text{ , for each } (i, j) \in L, \tag{5.10}$$

$$\bar{c}_{ij} \leq 0 \text{ , for each } (i, j) \in U. \tag{5.11}$$

These optimality conditions have a nice economic interpretation. We shall see a little later that if $\pi(1) = 0$, then equations (5.9) imply that $-\pi(j)$ denotes the length of the tree path in B from node 1 to node j. Then, $\bar{c}_{ij} = c_{ij} - \pi(i) + \pi(j)$ for a nonbasic arc (i, j) in L denotes the change in the cost of flow achieved by sending one unit of flow through the tree path from node 1 to node i, through the arc (i, j), and then returning the flow along the tree path from node j to node 1. The condition (5.10) implies that this circulation of flow is not profitable for any nonbasic arc in L. The condition (5.11) has a similar interpretation.

The network simplex algorithm maintains a feasible basis structure at each iteration and successively improves the basis structure until it becomes an optimum basic structure. The following algorithmic description specifies the essential steps of the procedure.

```
algorithm NETWORK SIMPLEX;
begin
    determine an initial basic feasible flow x and the corresponding
      basis structure (B, L, U);
    compute node potentials for this basis structure;
    while some arc violates the optimality conditions do
    begin
        select an entering arc (k, l) violating the optimality conditions;
        add arc (k, l) to the spanning tree corresponding to the basis forming a cycle
        and augment the maximum possible flow in this cycle;
        determine the leaving arc (p, q);
        perform a basis exchange and update node potentials;
    end;
end;
```

In the following discussion, we describe the various steps performed by the network simplex algorithm in greater detail.

Obtaining an Initial Basis Structure

Our connectedness assumption A5.2 provides one way of obtaining an initial basic feasible solution. We have assumed that for every node $j \in N - \{1\}$, the network contains arcs (1, j) and (j, 1) with sufficiently large costs and capacities. The initial basis B includes the arc (1, j) with flow -b(j) if $b(j) \leq 0$ and arc (j, 1) with flow b(j) if $b(j) > 0$. The set L consists of the remaining arcs, and the set U is empty. The node potentials for this basis are easily computed using (5.9), as we will see later.

Maintaining the Tree Structure

The specialized network simplex algorithm is possible because of the spanning tree property of the basis. The algorithm requires the tree to be represented so that the simplex algorithm can perform operations efficiently and update the representation quickly when the basis changes. We next describe one such tree representation.

We consider the tree as "hanging" from a specially designated node, called the *root*. We assume that node 1 is the root node. See Figure 5.1 for an example of the tree. We associate three indices with each node i in the tree: a *predecessor* index, *pred(i)*; a *depth* index, *depth(i)*; and a *thread* index, *thread(i)*. Each node i has a unique path connecting it

to the root. The predecessor index stores the first node in that path (other than node i) and the depth index stores the number of arcs in the path. For the root node these indices are zero. The Figure 5.1 shows an example of these indices. Note that by iteratively using the predecessor indices, we can enumerate the path from any node to the root node. We say that *pred(i)* is the *predecessor* of node i and i is a *successor* of node *pred(i)*. The *descendants* of a node i consist of the node i itself, its successors, successors of its successors, and so on. For example, the node set {5, 6, 7, 8, 9} contains the descendents of node 5 in Figure 5.1. A node with no successors is called a *leaf* node. In Figure 5.1, nodes 4, 7, 8, and 9 are leaf nodes.

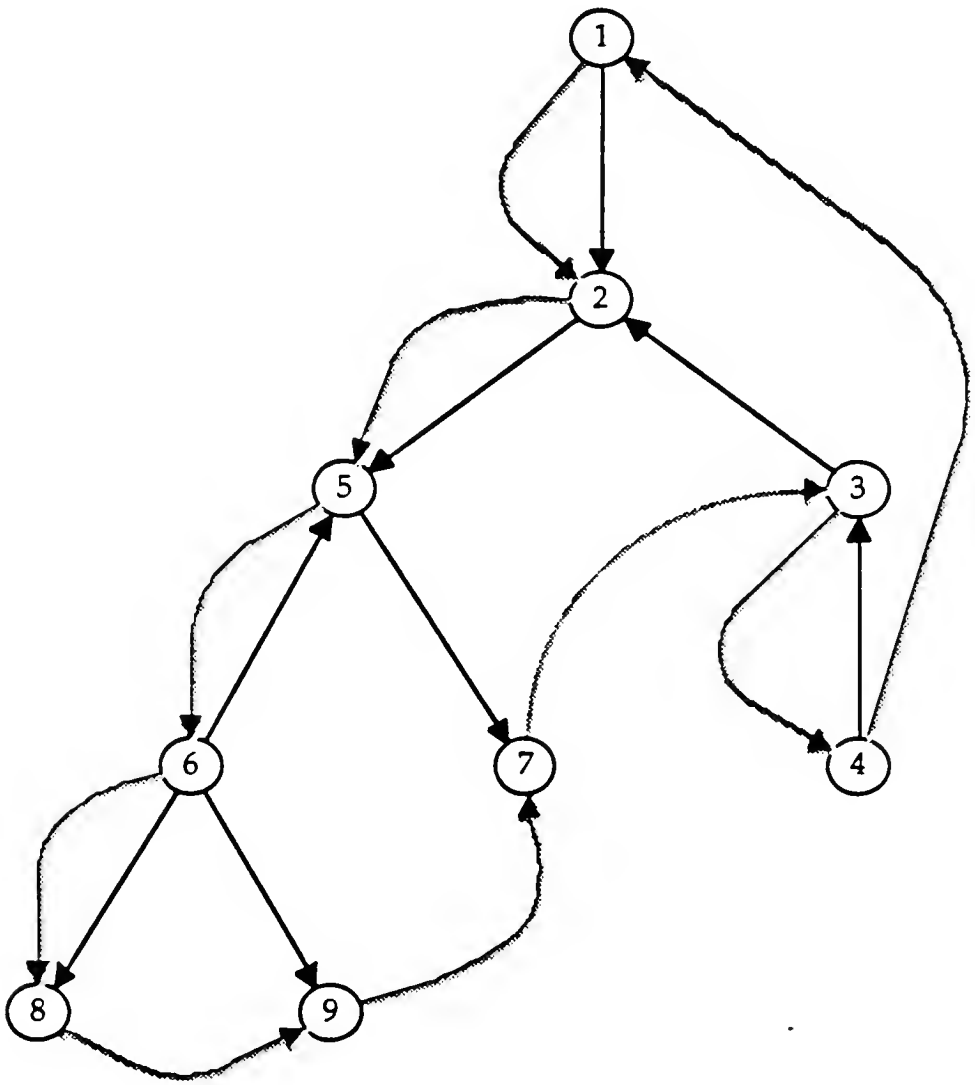
The thread indices define a *traversal* of the tree, a sequence of nodes that walks or threads its way through the nodes of the tree, starting at the root and visiting nodes in a "top to bottom" and "left to right" order, and then finally returning to the root. The thread indices can be formed by performing a depth first search of the tree as described in Section 1.5 and setting the thread of a node to be the node encountered after the node itself in this depth first search. For our example, this sequence would read 1-2-5-6-8-9-7-3-4-1 (see the dotted lines in Figure 5.1). For each node i, thread(i) specifies the next node in the traversal visited after node i. This traversal satisfies the following two properties: (i) the predecessor of each node appears in the sequence before the node itself; and (ii) the descendants of any node are consecutive elements in the traversal. The thread indices provide a particularly convenient means for visiting (or finding) all descendants of a node i: We simply follow the thread from node i, recording the nodes visited until the depth of the visited node becomes at least as large as node i. For example, starting at node 5, we visit nodes 6, 8, 9, and 7 in order, which are the descendants of node 5, and then visit node 3. Since node 3's depth equals that of node 5, we know that we have left the "descendant tree" lying below node 5. As we will see, finding the descendant tree of a node efficiently adds significantly to the efficiency of the simplex method.

The simplex method has two basic steps: (i) determining the node potentials of a given basis structure; and (ii) computing the arc flows for a given basis structure. We now describe how to perform these steps efficiently using the tree indices.

Computing Node Potentials and Flows for a Given Basis Structure

We first consider the problem of computing node potentials π for a given basis structure (B, L, U). We assume that $\pi(1) = 0$. Note that the value of one node potential

can be set arbitrarily since one constraint in (5.1b) is redundant. We compute the remaining node potentials using the conditions that $\bar{c}_{ij} = 0$ for each arc (i, j) in B. These conditions can alternatively be stated as



i	1	2	3	4	5	6	7	8	9
pred(i)	0	1	2	3	2	5	5	6	6
depth(i)	0	1	2	3	2	3	3	4	4
thread(i)	2	5	4	1	6	8	3	9	7

Figure 5.1. Example of a rooted tree and tree indices.

$$\pi(j) = \pi(i) - c_{ij}, \text{ for every arc } (i, j) \in B. \qquad (5.12)$$

The basic idea is to start at node 1 and fan out along the tree arcs using the thread indices to compute other node potentials. The traversal assures that whenever this fanning out procedure visits node j, it has already evaluated the potential of its predecessor, say node i; hence, the procedure can comput $\pi(j)$ using (5.12). The thread indices allow us to compute all node potentials in O(n) time using the following method.

```
procedure COMPUTE POTENTIALS;
begin
    π(1) : = 0;
    j : = thread(1);
    while j ≠ 1 do
    begin
        i : = pred(j);
        if (i, j) ∈ A then π(j) : = π(i) − c_ij;
        if (j, i) ∈ A then π(j) : = π(i) + c_ji;
        j : = thread(j);
    end;
end;
```

A similar procedure will permit us to compute flows on basic arcs for a given basis structure (B, L, U). We proceed, however, in the reverse order: start at the leaf node and move in toward the root using the predecessor indices, while computing flows on arcs encountered along the way. The following procedure accomplishes this task.

```
procedure COMPUTE FLOWS;
begin
    e(i) : = b(i) for all i ∈ N;
    let T be the basis tree;
    for each (i, j) ∈ U do
        set x_ij : = u_ij, subtract u_ij from e(i) and add u_ij to e(j);
    while T ≠ {1} do
    begin
        select a leaf node j in the subtree T;
        i : = pred(j);
        if (i, j) ∈ T then x_ij : = -e(j);
        else x_ji : = e(j);
        add e(j) to e(i);
        delete node j and the arc incident to it from T;
    end;
end;
```

One way of identifying leaf nodes in T is to select nodes in the reverse order of the thread indices. A simple procedure completes this task in O(n) time: push all the nodes into a stack in order of their appearance on the thread, and then take them out from the top one at a time. Note that in the thread traversal, each node appears prior to its descendants. Hence, the reverse thread traversal examines each node after examining its descendants.

Now consider the steps of the method. The arcs in the set U must carry flow equal to their capacity. Thus, we set $x_{ij} = u_{ij}$ for these arcs. This assignment creates an additional demand of u_{ij} units at node i and makes the same amount available at node j. This effect of setting $x_{ij} = u_{ij}$ explains the initial adjustments in the supply/demand of nodes. The manner for updating e(j) implies that each e(j) represents the sum of the adjusted supply/demand of nodes in the subtree hanging from node j. Since this subtree is connected to the rest of the tree only by the arc (i, j) (or (j, i)), this arc must carry -e(j) (or e(j)) units of flow to satisfy the adjusted supply/demand of nodes in the subtree.

The procedure Compute Flows essentially solves the system of equations $\mathbf{B}x = b$, in which **B** represents the columns in the node-arc incidence matrix **N** corresponding to the spanning tree T. Since **B** is a lower triangular matrix (see Theorem 2.6 in Section 2.3), it is possible to solve these equations by forward substitution, which is precisely

what the algorithm does. Similarly, the procedure Compute Potentials solves the system of equations $\pi B = c$ by back substitution.

Entering Arc

Two types of arcs are eligible to enter the basis: any nonbasic arc at its lower bound with a negative reduced cost or any nonbasic arc at its upper bound with a positive reduced cost, is eligible to enter the basis. These arcs violate condition (5.10) or (5.11). The method used for selecting an entering arc among these eligible arcs has a major effect on the performance of the simplex algorithm. An implementation that selects an arc that violates the optimality condition the most, i.e., has the largest value of $|\bar{c}_{ij}|$ among such arcs, might require the fewest number of iterations in practice, but must examine each arc at each iteration, which is very time-consuming. On the other hand, examining the arc list cyclically and selecting the *first* arc that violates the optimality condition would quickly find the entering arc, but might require a relatively large number of iterations due to the poor arc choice. One of the most successful implementations uses a *candidate list* approach that strikes an effective compromise between these two strategies. This approach also offers sufficient flexibility for fine tuning to special problem classes.

The algorithm maintains a candidate list of arcs violating the optimality conditions, selecting arcs in a two-phase procedure consisting of *major* iterations and *minor* iterations. In a major iteration, we construct the candidate list. We examine arcs emanating from nodes, one node at a time, adding to the candidate list the arcs emanating from node i (if any) that violate the optimality condition. We repeat this selection process for nodes i+1, i+2, ... until either we have examined all nodes or the list has reached its maximum allowable size. The next major iteration begins with the node where the previous major iteration ended. In other words, the algorithm examines nodes cyclically as it adds arcs emanating from them to the candidate list.

Once the algorithm has formed the candidate list in a major iteration, it performs minor iterations, scanning all candidate arcs and choosing a nonbasic arc from this list that violates the optimality condition the most to enter the basis. As we scan the arcs, we update the candidate list by removing those arcs that no longer violate the optimality conditions. Once the list becomes empty or we have reached a specified limit on the number of minor iterations to be performed at each major iteration, we rebuild the list with another major iteration.

Leaving Arc

Suppose we select the arc (k, *l*) as the entering arc. The addition of this arc to the basis B forms exactly one (undirected) cycle W, which is sometimes referred to as the *pivot cycle*. We define the orientation of W as the same as that of (k, *l*) if (k, *l*) $\in$ L, and opposite to the orientation of (k, *l*) if (k, *l*) $\in$ U. Let $\overline{W}$ and $\underline{W}$, respectively, denote the sets of arcs in W along and opposite to the cycle's orientation. Sending additional flow around the pivot cycle W in the direction of its orientation strictly decreases the cost of the current solution. We change the flow as much as possible until one of the arcs in the cycle W reaches its lower or upper bound; this arc leaves the basis. The maximum flow change δ_{ij} on an arc (i, j) $\in$ W that satisfies the flow bound constraints is

$$\delta_{ij} = \begin{cases} u_{ij} - x_{ij}, & \text{if } (i, j) \in \overline{W}, \\ x_{ij}, & \text{if } (i, j) \in \underline{W}. \end{cases}$$

We send $\delta = \min \{\delta_{ij} : (i, j) \in W\}$ units of flow around W, and select an arc (p, q) with $\delta_{pq} = \delta$ as the leaving arc. The crucial operation in this step is to identify the cycle W. If P(i) denotes the unique path in the basis from any node i to the root node, then this cycle consists of the arcs $\{(\{(k, l)\} \cup P(k) \cup P(l)) - (P(k) \cap P(l))\}$. In other words, W consists of the arc (k, *l*) and the disjoint portions of P(k) and P(*l*). Using predecessor indices alone permits us to identify the cycle W as follows. Start at node k and using predecessor indices trace the path from this node to the root and label all the nodes in this path. Repeat the same operation for node *l* until we encounter a node already labeled, say node w. Node w, which we might refer to as the *apex*, is the first common ancestor of nodes k and *l*. The cycle W contains the portions of the path P(k) and P(*l*) up to node w, along with the arc (k, *l*). This method is efficient, but it can be improved. It has the drawback of backtracking along some arcs that are not in W, namely, those in the portion of the path P(k) lying between the apex w and the root. The simultaneous use of depth and predecessor indices, as indicated in the following procedure, eliminates this extra work.

```
procedure IDENTIFY CYCLE;
begin
        i : = k and j : = l;
        while i ≠ j do
        begin
                if depth(i) > depth(j) then i : = pred(i)
                else if depth(j) > depth(i) then j : = pred(j)
                        else i : = pred(i) and j : = pred(j);
        end;
        w : = i;
end;
```

A simple modification of this procedure permits it to determine the flow δ that can be augmented along W as it determines the first common ancestor w of nodes k and *l*. Using predecessor indices to again traverse the cycle W, the algorithm can then update flows on arcs. The entire flow change operation takes O(n) time in the worst-case, but typically examines only a small subset of the nodes.

Basis Exchange

In the terminology of the simplex method, a basis exchange is a pivot operation. If $\delta = 0$, then the pivot is said to be *degenerate*; otherwise it is *nondegenerate*. A basis is called *degenerate* if flow on some basic arc equals its lower or upper bound, and *nondegenerate* otherwise. Observe that a degenerate pivot occurs only in a degenerate basis.

Each time the method exchanges an entering arc (k, *l*) for a leaving arc (p, q), it must update the basis structure. If the leaving arc is the same as the entering arc, which would happen when $\delta = u_{kl}$, the basis does not change. In this instance, the arc (k,*l*) merely moves from the set L to the set U, or vice versa. If the leaving arc differs from the entering arc, then more extensive changes are needed. In this instance, the arc (p, q) becomes a nonbasic arc at its lower or upper bound depending upon whether $x_{pq} = 0$ or $x_{pq} = u_{pq}$. Adding (k, *l*) and deleting (p, q) from the previous basis yields a new basis that is again a spanning tree. The node potentials also change and can be updated as follows. The deletion of the arc (p, q) from the previous basis partitions the set of nodes into two subtrees--one, T_1, containing the root node, and the other, T_2, not containing the root node. Note that the subtree T_2 hangs from node p or node q. The arc (k, *l*) has

one endpoint in T_1 and the other in T_2. As is easy to verify, the conditions $\pi(1) = 0$, and $c_{ij} - \pi(i) + \pi(j) = 0$ for all arcs in the new basis imply that the potentials of nodes in the subtree T_1 remain unchanged, and the potentials of nodes in the subtree T_2 change by a constant amount. If $k \in T_1$ and $l \in T_2$, then all the node potentials in T_2 change by $-\bar{c}_{kl}$; if $l \in T_1$ and $k \in T_2$, they change by the amount $\bar{c}_{kl}$. The following method, using the thread and depth indices, updates the node potentials quickly.

```
procedure UPDATE POTENTIALS;
begin
    if q ∈ T2 then y := q else y := p;
    if k ∈ T1 then change := - c̄kl else change := c̄kl;
    π(y) := π(y) + change;
    z := thread(y);
    while depth(z) < depth(y) do
    begin
        π(z) := π(z) + change;
        z := thread(z);
    end;
end;
```

The final step in the basis exchange is to update various indices. This step is rather involved and we refer the reader to the reference material cited in Section 6.4 for the details. We do note, however, that it is possible to update the tree indices in O(n) time.

Termination

The network simplex algorithm, as just described, moves from one basis structure to another until it obtains a basis structure that satisfies the optimality conditions (5.9)–(5.11). It is easy to show that the algorithm terminates in a finite number of steps if each pivot operation is nondegenerate. Recall that $|\bar{c}_{kl}|$ represents the net decrease in the cost per unit flow sent around the cycle W. During a nondegenerate pivot (in which $\delta > 0$), the new basis structure has a cost that is $\delta|\bar{c}_{kl}|$ units lower than the previous basis structure. Since there are a finite number of basis structures and every basis structure has a unique associated cost, the network simplex algorithm will terminate finitely assuming nondegeneracy. Degenerate pivots, however, pose theoretical difficulties that we address next.

Strongly Feasible Bases

The network simplex algorithm does not necessarily terminate in a finite number of iterations unless we impose an additional restriction on the choice of entering and leaving arcs. Researchers have constructed very small network examples for which poor choices lead to *cycling*, i.e., an infinite repetitive sequence of degenerate pivots. Degeneracy in network problems is not only a theoretical issue, but also a practical one. Computational studies have shown that as many as 90% of the pivot operations in common networks can be degenerate. As we show next, by maintaining a special type of basis, called a *strongly feasible basis*, the simplex algorithm terminates finitely; moreover, it runs faster in practice as well.

Let (B, L, U) be a basis structure of the minimum cost flow problem with integral data. As earlier, we conceive of a basis tree as a tree hanging from the root node. The tree arcs either are *upward pointing* (towards the root) or are *downward pointing* (away from the root). We say that a basis structure (B, L, U) is *strongly feasible* if we can send a positive amount of flow from any node in the tree to the root along arcs in the tree without violating any of the flow bounds. See Figure 5.2 for an example of a strongly feasible basis. Observe that this definition implies that no upward pointing arc can be at its upper bound and no downward pointing arc can be at its lower bound.

The *perturbation technique* is a well-known method for avoiding cycling in the simplex algorithm for linear programming. This technique slightly perturbs the right-hand-side vector so that every feasible basis is nondegenerate and so that it is easy to convert an optimum solution of the perturbed problem to an optimum solution of the original problem. We show that a particular perturbation technique for the network simplex method is equivalent to the combinatorial rule known as the *strongly feasible basis technique.*

The minimum cost flow problem can be perturbed by changing the supply/demand vector b to $b+\varepsilon$. We say that $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ is a feasible perturbation if it satisfies the following conditions:

(i) $\varepsilon_i > 0$ for all $i = 2, 3, \ldots, n$;

(ii) $\sum_{i=2}^{n} \varepsilon_i < 1$; and

(iii) $\varepsilon_1 = -\sum_{i=2}^{n} \varepsilon_i$.

One possible choice for a feasible perturbation is $\varepsilon_i = 1/n$ for i = 2, ... , n (and thus $\varepsilon_1 = -(n-1)/n$). Another choice is $\varepsilon_i = \alpha^i$ for i = 2, ... , n, with α chosen as a very small positive number. The perturbation changes the flow on basic arcs. The justification procedure we gave for the Compute-Flows, earlier in this section, implies that perturbation of b by ε changes the flow on basic arcs in the following manner:

1. If (i, j) is a downward pointing arc of tree B and D(j) is the set of descendants of node j, then the perturbation decreases the flow in arc (i, j) by $\sum_{k \in D(j)} \varepsilon_k$. Since $0 < \sum_{k \in D(j)} \varepsilon_k < 1$, the resulting flow is nonintegral and thus nonzero.

2. If (i, j) is an upward pointing arc of tree B and D(i) is the set of descendants of node i, then the perturbation increases the flow in arc (i, j) by $\sum_{k \in D(i)} \varepsilon_k$. Since $0 < \sum_{k \in D(i)} \varepsilon_k < 1$, the resulting flow is nonintegral and thus nonzero.

Theorem 5.2. For any basis structure (B, L, U) of the minimum cost flow problem, the following statements are equivalent:

(i) *(B, L, U) is strongly feasible.*

(ii) *No upward pointing arc of the basis is at its upper bound and no downward pointing arc of the basis is at its lower bound.*

(iii) *(B, L, U) is feasible if we replace b by b+ε, for any feasible perturbation ε .*

(iv) *(B, L, U) is feasible if we replace b by b+ε, for the perturbation ε = (-(n-1)/n, 1/n, 1/n, ... , 1/n).*

Proof. (i) $\Rightarrow$ (ii). Suppose an upward pointing arc (i, j) is at its upper bound. Then node i cannot send any flow to the root, violating the definition of a strongly feasible basis. For the same reason, no downward pointing arc can be at its lower bound.

(ii) $\Rightarrow$ (iii). Suppose that (ii) is true. As noted earlier, perturbation increases the flow on an upward pointing arc by an amount strictly between 0 and 1. Since the flow on an upward pointing arc is integral and strictly less than its (integral) upper bound, the perturbed solution remains feasible. Similar reasoning shows that after we have perturbed the problem, downward pointing arcs also remain feasible.

(iii) $\Rightarrow$ (iv). Follows directly because $\varepsilon = (-(n-1)/n, 1/n, 1/n, \ldots, 1/n)$ is a feasible perturbation.

(iv) $\Rightarrow$ (i). Consider the feasible basis structure (B, L, U) of the perturbed problem. Each arc in the basis B has a positive nonintegral flow. Consider the same basis tree for the original problem. If we remove the perturbation (i.e., replace $b + \varepsilon$ by b), flows on the downward pointing arcs increase, flows on the upward pointing arcs decrease, and the resulting flows are integral. Consequently, $x_{ij} > 0$ for downward pointing arcs, $x_{ij} < u_{ij}$ for upward pointing arcs, and (B, L, U) is strongly feasible for the original problem. ■

This theorem shows that maintaining a strongly feasible basis is equivalent to applying the ordinary simplex algorithm to the perturbed problem. This result implies that both approaches obtain exactly the same sequence of basis structures if they use the same rule to select the entering arcs. As a corollary, this equivalence shows that any implementation of the simplex algorithm that maintains a strongly feasible basis performs at most nmCU pivots. To establish this result, consider the perturbed problem with the perturbation $\varepsilon = (-(n-1)/n, 1/n, 1/n, \ldots, 1/n)$. With this perturbation, the flow on every arc is a multiple of 1/n. Consequently, every pivot operation augments at least 1/n units of flow and therefore decreases the objective function value by at least 1/n units. Since mCU is an upper bound on the objective function value of the starting solution and zero is a lower bound on the minimum objective function value, the algorithm will terminate in at most nmCU iterations. Therefore, any implementation of the simplex algorithm that maintains a strongly feasible basis runs in pseudopolynomial time.

We can thus maintain strong feasibility by perturbing b by a suitable perturbation ε. However, there is no need to actually perform the perturbation. Instead, we can maintain strong feasibility using a "combinatorial rule" that is equivalent to applying the original simplex method after we have imposed the perturbation. Even though this rule permits degenerate pivots, it is guaranteed to converge. Figure 5.2 will illustrate our discussion of this method.

Combinatorial Version of Perturbation

The network simplex algorithm starts with a strongly feasible basis. The method described earlier to construct the initial basis always gives such a basis. The algorithm selects the leaving arc in a degenerate pivot carefully so that the next basis is also

feasible. Suppose that the entering arc (k, *l*) is at its lower bound and the apex w is the first common ancestor of nodes k and *l*. Let W be the cycle formed by adding arc (k, *l*) to the basis tree. We define the orientation of the cycle as the same as that of arc (k, *l*). After updating the flow, the algorithm identifies the *blocking arcs*, i.e., those arcs (i, j) in the cycle W that satisfy $\delta_{ij} = \delta$. If the blocking arc is unique, then it leaves the basis. If the cycle contains more than one blocking arc, then the next basis will be degenerate; i.e., some basic arcs will be at their lower or upper bounds. In this case, the algorithm selects the leaving arc in accordance with the following rule:

Combinatorial Pivot Rule. *When introducing an arc into the basis for the network simplex method, select the leaving arc as the last blocking arc, say arc (p, q), encountered in traversing the pivot cycle W along its orientation starting at the apex w.*

We next show that this rule guarantees that the next basis is strongly feasible. To do so, we show that in this basis every node in the cycle W can send positive flow to the root node. Notice that since the previous basis was strongly feasible, every node could send positive flow to the root node. Let W_1 be the segment of the cycle W between the apex w and arc (p, q), when we traverse the cycle along its orientation. Further, let $W_2 = W - W_1 - \{(p, q)\}$. Define the orientation of segments W_1 and W_2 to be compatable with the orientation of W. See Figure 5.2 for an illustration of the segments W_1 and W_2 for our example. Since arc (p, q) is the last blocking arc in W, no arc in W_2 is blocking and every node contained in the segment W_2 can send positive flow to the root along the orientation of W_2 and via node w. Now consider nodes contained in the segment W_1. We distinguish two cases. If the current pivot was a *nondegenerate* pivot, then the pivot augmented a positive amount of flow along the arcs in W_1; hence, every node in the segment W_1 can augment flow back to the root opposite to the orientation of W_1 and via node w. If the current pivot was a *degenerate* pivot, then W_1 must be contained in the segment of W between node w and node k, because by the property of strong feasibility, every node on the path from node *l* to node w can send a positive amount of flow to the root before the pivot and, thus, no arc on this path can be a blocking arc in a degenerate pivot. Now observe that before the pivot, every node in W_1 could send positive flow to the root and, therefore, since the pivot does not change flow values, every node in W_1 must be able to send positive flow to the root after the pivot as well. This conclusion completes the proof that the next basis is strongly feasible.

We now study the effect of the basis change on node potentials during a degenerate pivot. Since arc (k, *l*) enters the basis at its lower bound, $\bar{c}_{kl} < 0$. The leaving arc belongs to the path from node k to node w. Hence, node k lies in the subtree T_2 and

the potentials of all nodes in T_2 change by the amount $-\bar{c}_{kl} > 0$. Consequently, this degenerate pivot strictly increases the sum of all node potentials (which by our prior assumptions is integral). Since the sum of all node potentials is bounded from below, the number of successive degenerate pivots is finite.

So far we have assumed that the entering arc is at its lower bound. If the entering arc (k, *l*) is at its upper bound, then we define the orientation of the cycle W as opposite to the orientation of arc (k, *l*). The criteria to select the leaving arc remains unchanged--the leaving arc is the last blocking arc encountered in traversing W along its orientation starting at node w. In this case, node *l* is contained in the subtree T_2 and, thus, after the pivot all nodes in T_2 again increase by the amount $-\bar{c}_{kl}$; consequently, the pivot again increases the sum of the node potentials.

Complexity Results

The strongly feasible basis technique implies some nice theoretical results about the network simplex algorithm implemented using Dantzig's pivot rule, i.e., pivoting in the arc that most violates the optimality conditions (that is, the arc (k, *l*) with the largest value of $|\bar{c}_{kl}|$ among all arcs that violate the optimality conditions). This technique also yields polynomial time simplex algorithms for the shortest path and assignment problems.

We have already shown that any version of the network simplex algorithm that maintains a strongly feasible basis performs O(nmCU) pivots. Using Dantzig's pivot rule and geometric improvement arguments, we can reduce the number of pivots to O(nmU log H), with H defined as H = mCU. As earlier, we consider the perturbed problem with perturbation $\varepsilon = (-(n-1)/n, 1/n, 1/n, \ldots, 1/n)$. Let z^k denote the objective function value of the perturbed minimum cost flow problem at the k-th iteration of the simplex algorithm, x denote the current flow, and (B, L, U) denote the current basis structure. Let $\Delta > 0$ denote the maximum violation of the optimality condition of any nonbasic arc. If the algorithm next pivots in a nonbasic arc corresponding to the maximum violation, then the objective function value decreases by at least Δ/n units. Hence,

$$z^k - z^{k+1} \geq \Delta/n. \tag{5.13}$$

We now need an upper bound on the total possible improvement in the objective function after the k-th iteration. It is easy to show that

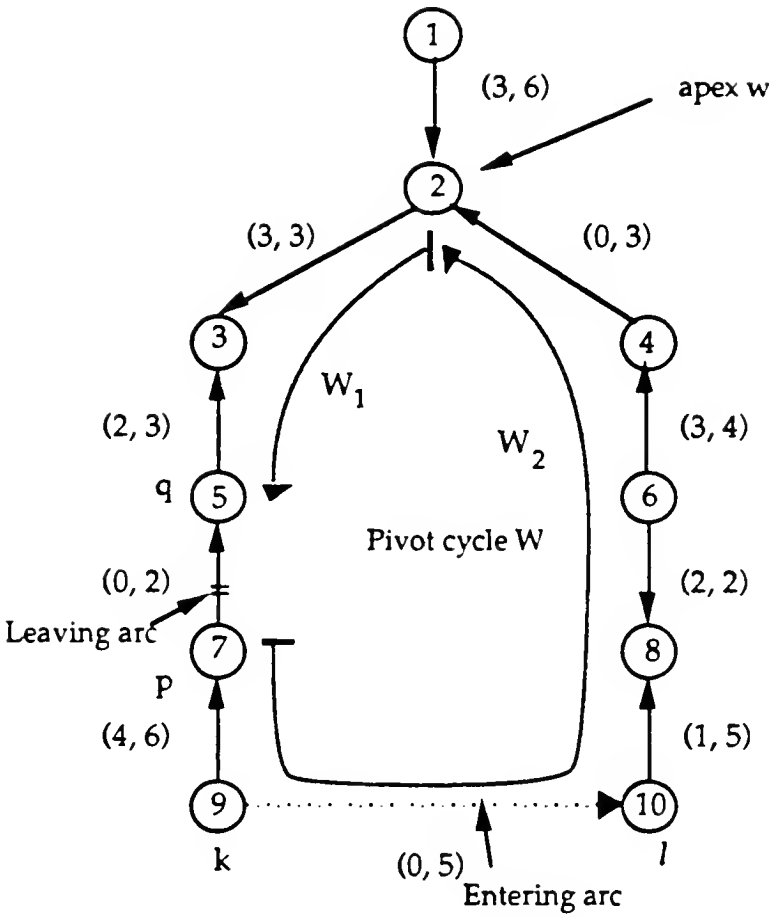


Figure 5.2. A strongly feasible basis. The figure shows the flows and capacities represented as (x_{ij}, u_{ij}). The entering arc is (9, 10); the blocking arcs are (2, 3) and (7, 5); and the leaving arc is (7, 5). This pivot is a degenerate pivot. The segments W_1 and W_2 are as shown.

$$\sum_{(i,j)\in A} \bar{c}_{ij} x_{ij} = \sum_{(i,j)\in A} c_{ij} x_{ij} - \sum_{i\in N} \pi(i)\, b(i).$$

Since the rightmost term in this expression is a constant for fixed values of the node potentials, the total improvement with respect to the objective function $\sum_{(i,j)\in A} c_{ij} x_{ij}$ is equal to the total improvement with respect to the objective function $\sum_{(i,j)\in A} \bar{c}_{ij} x_{ij}$. Further, the total improvement in the objective function $\sum_{(i,j)\in A} \bar{c}_{ij} x_{ij}$ is bounded by the total improvement in the following relaxed problem:

$$\text{minimize} \sum_{(i,j)\in A} \bar{c}_{ij} x_{ij}, \tag{514a}$$

subject to

$$0 \le x_{ij} \le u_{ij}, \text{ for all } (i,j) \in A. \tag{5.14b}$$

For a given basis structure (B, L, U), we construct an optimum solution of (5.14) by setting $x_{ij} = u_{ij}$ for all arcs $(i,j) \in L$ with $\bar{c}_{ij} < 0$, by setting $x_{ij} = 0$ for all arcs $(i,j) \in U$ with $\bar{c}_{ij} > 0$, and by leaving the flow on the basic arcs unchanged. This readjustment of flow decreases the objective function by at most $m\Delta U$. We have thus shown that

$$z^k - z^* \le m\Delta U. \tag{5.15}$$

Combining (5.13) and (5.15) we obtain

$$(z^k - z^{k+1}) \ge \frac{1}{nmU}(z^k - z^*).$$

By Lemma 1.1, if $H = mCU$, the network simplex algorithm terminates in O(nmU log W) iterations. We summarize our discussion as follows.

Theorem 5.3. The network simplex algorithm that maintains a strongly feasible basis and uses Dantzig's pivot rule performs O(nmU log H) pivots. ■

This result gives polynomial time bounds for the shortest path and assignment problems since both can be formulated as minimum cost flow problems with U = n and U = 1 respectively. In fact, it is possible to modify the algorithm and use the previous arguments to show that the simplex algorithm solves these problems in $O(n^2 \log C)$ pivots and runs in O(nm log C) total time. These results can be found in the references cited in Section 6.4.

5.7 Right-Hand-Side Scaling Algorithm

Scaling techniques are among the most effective algorithmic strategies for designing polynomial time algorithms for the minimum cost flow problem. In this section, we describe an algorithm based on a right-hand-side scaling (RHS-scaling) technique. The next two sections present polynomial time algorithms based upon cost scaling, and simultaneous right-hand-side and cost scaling.

The RHS-scaling algorithm is an improved version of the successive shortest path algorithm. The inherent drawback in the successive shortest path algorithm is that augmentations may carry relatively small amounts of flow, resulting in a fairly large number of augmentations in the worst case. The RHS-scaling algorithm guarantees that each augmentation carries *sufficiently large* flow and thereby reduces the number of augmentations substantially. We shall illustrate RHS-scaling on the uncapacitated minimum cost flow problem, i.e., a problem with $u_{ij} = \infty$ for each (i, j) $\in$ A. This algorithm can be applied to the capacitated minimum cost flow problem after it has been converted into an uncapacitated problem (as described in Section 2.4).

The algorithm uses the pseudoflow x and the imbalances e(i) as defined in Section 5.4. It performs a number of scaling phases. Much as we did in the excess scaling algorithm for the maximum flow problem, we let Δ be the least power of 2 satisfying either (i) e(i) < 2Δ for all i, or (ii) e(i) > -2Δ for all i, but not necessarily both. Initially, $\Delta = 2^{\lceil \log U \rceil}$. This definition implies that the sum of excesses (whose magnitude is equal to the sum of deficits) is bounded by $2n\Delta$. Let $S(\Delta) = \{ i : e(i) \geq \Delta \}$ and let $T(\Delta) = \{ j : e(j) \leq -\Delta \}$. Then at the beginning of the Δ-scaling phase, either $S(2\Delta) = \emptyset$ or $T(2\Delta) = \emptyset$. In the given Δ-scaling phase, we perform a number of augmentations, each from a node i $\in$ S(Δ) to a node j $\in$ T(Δ), and each of these augmentations carries Δ units of flow. The definition of Δ implies that within n augmentations the algorithm will decrease Δ by a factor of at least 2. At this point, we begin a new scaling phase. Hence, within O(log U) scaling

phase, $\Delta < 1$. By the integrality of data, all imbalances are now zero and the algorithm has found an optimum flow.

The driving force behind this scaling technique is an invariant property (which we will prove later) that each arc flow in the Δ-scaling phase is a multiple of Δ. This flow invariant property and the connectedness assumption (A5.2) ensure that we can always send Δ units of flow from a node in $S(\Delta)$ to a node in $T(\Delta)$. The following algorithmic description is a formal statement of the RHS-scaling algorithm.

```
algorithm RHS-SCALING;
begin
    x := 0, e := b;
    let π be the shortest path distances in G(0);
    Δ := 2^⌈log U⌉;
    while the network contains a node with nonzero imbalance do
    begin
        S(Δ) := { i ∈ N : e(i) ≥ Δ };
        T(Δ) := { i ∈ N : e(i) ≤ -Δ };
        while S(Δ) ≠ ø and T(Δ) ≠ ø do
        begin
            select a node k ∈ S(Δ) and a node l ∈ T(Δ);
            determine shortest path distances d from node k to all other nodes
                in the residual network G(x) with respect to the reduced costs
                c̄ij;
            let P denote the shortest path from node k to node l;
            update π := π - d;
            augment Δ units of flow along the path P;
            update x, S(Δ) and T(Δ);
        end;
        Δ := Δ/2;
    end;
end;
```

The RHS-scaling algorithm correctly solves the problem because during the Δ-scaling phase, it is able to send Δ units of flow on the shortest path from a node $k \in S(\Delta)$ to a node $l \in T(\Delta)$. This fact follows from the following result.

Lemma 5.2. The residual capacities of arcs in the residual network are always integer multiples of Δ.

Proof. We use induction on the number of augmentations and scaling phases. The initial residual capacities are a multiple of Δ because they are either 0 or ∞. Each augmentation changes the residual capacities by 0 or Δ units and preserves the inductive hypothesis. A decrease in the scale factor by a factor of 2 also preserves the inductive hypothesis. This result implies the conclusion of the lemma. ■

Let S(n, m, C) denote the time to solve a shortest path problem on a network with nonnegative arc lengths.

Theorem 5.4. The RHS-scaling algorithm correctly computes a minimum cost flow and performs O(n log U) augmentations and consequently solves the minimum cost flow problem in O(n log U S(n, m, C)) time.

Proof. The RHS-scaling algorithm is a special case of the successive shortest path algorithm and thus terminates with a minimum cost flow. We show that the algorithm performs at most n augmentations per scaling phase. Since the algorithm requires $1+\lceil \log U \rceil$ scaling phases, this fact would imply the conclusion of the theorem. At the beginning of the Δ-scaling phase, either $S(2\Delta) = \emptyset$ or $T(2\Delta) = \emptyset$. We consider the case when $S(2\Delta) = \emptyset$. A similar proof applies when $T(2\Delta) = \emptyset$. At the beginning of the scaling phase, $|S(\Delta)| \leq n$. Observe that $\Delta \leq e(i) < 2\Delta$ for each node $i \in S(\Delta)$. Each augmentation starts at a node in $S(\Delta)$, ends at a node with a deficit, and carries Δ units of flow; therefore, it decreases $|S(\Delta)|$ by one. Consequently, each scaling phase can perform at most n augmentations. ■

Applying the scaling algorithm directly to the capacitated minimum cost flow problem introduces some subtlety, because Lemma 5.2 does not apply for this situation. The inductive hypothesis fails to be true initially since the residual capacities are 0 or u_{ij}. As we noted previously, one method of solving the capacitated minimum cost flow problem is to first transform the capacitated problem to an uncapacitated one using the technique described in Section 2.4. We then apply the RHS-scaling algorithm on the transformed network. The transformed network contains n+m nodes, and each scaling phase performs at most n+m augmentations. The shortest path problem on the transformed problem can be solved (using some clever techniques) in S(n, m, C) time. Consequently, the RHS-scaling algorithm solves the capacitated minimum cost flow problem in O(m log U S(n, m, C)) time. A recently developed modest variation of the RHS-scaling algorithm solves the capacitated minimum cost flow problem in O(m log n

$(m + n \log n))$ time. This method is currently the best *strongly* polynomial-time algorithm for solving the minimum cost flow problem.

5.8. Cost Scaling Algorithm

We now describe a cost scaling algorithm for the minimum cost flow problem. This algorithm can be viewed as a generalization of the preflow-push algorithm for the maximum flow problem.

This algorithm relies on the concept of *approximate optimality*. A flow x is said to be *ε -optimal* for some $\varepsilon > 0$ if x together with some node potentials π satisfy the following conditions.

C5.7 (Primal feasibility) x is feasible.
C5.8. (ε -Dual feasibility) $\bar{c}_{ij} \geq -\varepsilon$ for each arc (i, j) in the residual network G(x).

We refer to these conditions as the *ε -optimality* conditions. These conditions are a relaxation of the original optimality conditions and reduce to C5.5 and C5.6 when ε is 0. The ε -optimality conditions permit $-\varepsilon \leq \bar{c}_{ij} < 0$ for an arc (i, j) at its lower bound and $\varepsilon \geq \bar{c}_{ij} > 0$ for an arc (i, j) at its upper bound, which is a relaxation of the usual optimality conditions. The following facts are useful for analysing the cost scaling algorithm.

Lemma 5.3. Any feasible flow is ε -optimal for $\varepsilon \geq C$. Any ε -optimal feasible flow for $\varepsilon < 1/n$ is an optimum flow.

Proof. Clearly, any feasible flow with zero node potentials satisfies C5.8 for $\varepsilon \geq C$. Now consider an ε-optimal flow with $\varepsilon < 1/n$. The ε-dual feasibility conditions imply that for any directed cycle W in the residual network, $\sum_{(i,j) \in W} c_{ij} = \sum_{(i,j) \in W} \bar{c}_{ij} \geq -n\varepsilon > -1$. Since all arc costs are integral, this result implies that $\sum_{(i,j) \in W} c_{ij} \geq 0$. Hence, the residual network contains no negative cost cycle and from Theorem 5.1 the flow is optimum. ■

The cost scaling algorithm treats ε as a parameter and iteratively obtains ε-optimal flows for successively smaller values of ε. Initially $\varepsilon = C$, and finally $\varepsilon < 1/n$. The algorithm performs cost scaling phases by repeatedly applying an *Improve-Approximation* procedure that transforms an ε-optimal flow into an ε/2-optimal flow. After $1+\lceil \log nC \rceil$

cost scaling phases, $\varepsilon < 1/n$ and the algorithm terminates with an optimum flow. More formally, we can state the algorithm as follows.

```
algorithm COST SCALING;
begin
    π := 0 and ε := C;
    let x be any feasible flow;
    while ε ≥ 1/n do
    begin
        IMPROVE-APPROXIMATION-I(ε, x, π);
        ε := ε/2 ;
    end;
    x is an optimum flow for the minimum cost flow problem;
end;
```

The Improve-Approximation procedure transforms an ε-optimal flow into an $\varepsilon/2$-optimal flow. It does so by (i) first converting an ε-optimal flow into an 0-optimal pseudoflow (a pseudoflow x is called ε-optimal if it satisfies the ε-dual feasibility conditions C5.8), and then (ii) gradually converting the pseudoflow into a flow while always maintaining the $\varepsilon/2$-dual feasibility conditions. We call a node i with $e(i) > 0$ *active* and call an arc (i, j) in the residual network *admissible* if $-\varepsilon/2 \le \bar{c}_{ij} < 0$. The basic operations are selecting active nodes and pushing flows on admissible arcs. We shall see later that pushing flows on admissible arcs preserves the $\varepsilon/2$-dual feasibility conditions. The Improve-Approximation procedure uses the following subroutine.

```
procedure PUSH/RELABEL(i);
begin
    if G(x) contains an admissible arc (i, j) then
        push δ := min { e(i), r_ij } units of flow from node i to node j ;
    else π(i) := π(i) + ε/2 + min { c_ij : (i, j) ∈ A(i) and r_ij > 0};
end;
```

Recall that r_{ij} denotes the residual capacity of an arc (i, j) in G(x). As in our earlier discussion of preflow-push algorithms for the maximum flow problem, if $\delta = r_{ij}$, then we refer to the push as *saturating*; otherwise it is *nonsaturating*. We also refer to the updating of the potential of a node as a *relabel* operation. The purpose of a relabel operation is to create new admissible arcs. Moreover, we use the same data structure as

used in the maximum flow algorithms to identify admissible arcs. For each node i, we maintain a *current arc* (i, j) which is the current candidate for pushing flow out of node i. The current arc is found by sequentially scanning the arc list A(i).

The following generic version of the Improve-Approximation procedure summarizes its essential operations.

```
procedure IMPROVE-APPROXIMATION-I(ε, x, π);
begin
    if c̄ij > 0 then xij := 0
    else if c̄ij < 0 then xij := uij ;
    compute node imbalances;
    while the network contains an active node do
    begin
        select an active node i;
        PUSH/RELABEL(i);
    end;
end;
```

The correctness of this procedure rests on the following result.

Lemma 5.4. The Improve-Approximation procedure always maintains ε/2-optimality of the pseudoflow, and at termination yields an ε/2-optimal flow.

Proof. This proof is similar to that of Lemma 4.1. At the beginning of the procedure, the algorithm adjusts the flows on arcs to obtain an ε/2-pseudoflow (in fact, it is a 0-optimal pseudoflow). We use induction on the number of push/relabel steps to show that the algorithm preserves ε/2-optimality of the pseudoflow. Pushing flow on arc (i, j) might add its reversal (j, i) to the residual network. But since $-\varepsilon/2 \le \bar{c}_{ij} < 0$ (by the criteria of admissibility), $\bar{c}_{ji} > 0$ and the condition C5.8 is satisfied for any value of $\varepsilon > 0$. The algorithm relabels node i when $\bar{c}_{ij} \ge 0$ for every arc (i, j) in the residual network. By our rule for increasing potentials, after we increase $\pi(i)$ by $\varepsilon/2 + \min\{\bar{c}_{ij} : (i, j) \in A(i) \text{ and } r_{ij} > 0\}$ units, the reduced cost of every arc (i, j) with $r_{ij} > 0$ still satisfies $\bar{c}_{ij} \ge -\varepsilon/2$. In addition, increasing $\pi(i)$ maintains the condition $\bar{c}_{ki} \ge -\varepsilon/2$ for all arc (k,i) in the residual network. Therefore, the procedure preserves ε/2-optimality of the pseudoflow throughout and, at termination, yields an ε/2-optimal flow. ■

We next analyze the complexity of the Improve-Approximation procedure. We will show that the complexity of the generic version is $O(n^2m)$ and then describe a specialized version running in time $O(n^3)$. These time bounds are comparable to those of the preflow-push algorithms for the maximum flow problem.

Lemma 5.5. No node potential increases more than 3n times during an execution of the Improve-Approximation procedure.

Proof. Let x be the current $\varepsilon/2$-optimal pseudoflow and x' be the ε-optimal flow at the end of the previous cost scaling phase. Let π and π' be the node potentials corresponding to the pseudoflow x and the flow x' repectively. It is possible to show, using a variation of the flow decomposition properties discussed in Section 2.1, that for every node v with positive imbalance in x there exists a node w with negative imbalance in x and a path P satisyfing the properties that (i) P is an augmenting path with respect to x, and (ii) its reversal $\overline{P}$ is an augmenting path with respect to x'. This fact in terms of the residual networks implies that there exists a sequence of nodes $v = v_0, v_1, \ldots, v_l = w$ with the property that $P = v_0 - v_1 - \ldots - v_l$ is a path in G(x) and its reversal $\overline{P} = v_l - v_{l-1} - \ldots - v_1$ is a path in G(x'). Applying the $\varepsilon/2$- optimality conditions to arcs on the path P in G(x), we obtain $\sum_{(i,j) \in P} \bar{c}_{ij} \geq -l(\varepsilon/2)$. Alternatively,

$$\pi(v) \leq \pi(w) + l(\varepsilon/2) + \sum_{(i,j) \in P} c_{ij}. \tag{5.16}$$

Applying the ε - optimality conditions to arcs on the path $\overline{P}$ in G(x'), we obtain

$$\pi'(w) \leq \pi'(v) + l\varepsilon + \sum_{(j,i) \in \overline{P}} c_{ji} = \pi'(v) + l\varepsilon - \sum_{(i,j) \in P} c_{ij}. \tag{5.17}$$

Combining (5.16) and (5.17) gives

$$\pi(v) \leq \pi'(v) + (\pi(w) - \pi'(w)) + (3/2)l\varepsilon. \tag{5.18}$$

Now we use the facts that (i) $\pi(w) = \pi'(w)$ (the potentials of a node with a negative imbalance does not change because the algorithm never selects it for push/relabel), (ii) $l \leq n$, and (iii) each increase in potential increases $\pi(v)$ by at least $\varepsilon/2$ units. The lemma is now immediate. ■

Lemma 5.6. The Improve-Approximation procedure performs O(nm) saturating pushes.

Proof. This proof is similar to that of Lemma 4.5 and essentially amounts to showing that between two consecutive saturations of an arc (i, j), the potentials of both the nodes i and j increase at least once. Since any node potential increases O(n) times, the algorithm also saturates any arc O(n) times resulting in O(nm) total saturating pushes. ■

To bound the number of nonsaturating pushes, we need one more result. We define the *admissible network* as the network consisting solely of admissible arcs. The following result is crucial to analyse the complexity of the cost scaling algorithms.

Lemma 5.7. The admissible network is acyclic throughout the cost scaling algorithms.

Proof. We establish this result by an induction argument applied to the number of pushes and relabels. The result is true at the beginning of each cost scaling phase because the pseudoflow is 0-optimal and the network contains no admissible arc. We always push flow on an arc (i, j) with $\bar{c}_{ij} < 0$; hence, if the algorithm adds its reversal (j, i) to the residual network, then $\bar{c}_{ji} > 0$. Thus pushes do not create new admissible arcs and preserve the inductive hypothesis. A relabel operation at node i may create new admissible arcs (i, j), but it also deletes all admissible arcs (k, i). The latter result is true because for any arc (k, i), $\bar{c}_{ki} \geq -\varepsilon/2$ before a relabel operation, and $\bar{c}_{ki} \geq 0$ after the relabel operation since the relabel operation increases $\pi(i)$ by at least $\varepsilon/2$ units. Therefore the algorithm can create no directed cycles. ■

Lemma 5.8. The Improve-Approximation procedure performs $O(n^2m)$ nonsaturating pushes.

Proof (Sketch). Let g(i) be the number of nodes that are reachable from node i in the admissible network and let the potential function $F = \sum_{i \text{ active}} g(i)$. The proof amounts to showing that a relabel operation or a saturating push can increase F by at most n units and each nonsaturating push decreases F by at least 1 unit. Since the algorithm performs at most $3n^2$ relabel operations and O(nm) saturation pushes, by Lemmas 5.5 and 5.6, these observations yield a bound of $O(n^2m)$ on the number of nonsaturating pushes. ■

As in the maximum flow algorithm, the bottleneck operation in the Improve-Approximation procedure is the nonsaturating pushes, which take $O(n^2m)$ time. The algorithm takes O(nm) time to perform saturating pushes, and the same time to scan arcs while identifying admissible arcs. Since the cost scaling algorithm calls Improve-Approximation $1+\lceil \log nC \rceil$ times, we obtain the following result.

Theorem 5.5. The generic cost scaling algorithm runs in $O(n^2 m \log nC)$ time. ■

The cost scaling algorithm illustrates an important connection between the maximum flow and the minimum cost flow problems. Solving an Improve-Approximation problem is very similar to solving a maximum flow problem. Just as in the generic preflow-push algorithm for the maximum flow problem, the bottleneck operation is the number of nonsaturating pushes. Researchers have suggested improvements based on examining nodes in some specific order, or using clever data structures. We describe one such improvement , called the *wave algorithm.*

The wave algorithm is the same as the Improve-Approximation procedure, but it selects active nodes for the push/relabel step in a specific order. The algorithm uses the acyclicity of the admissible network. As is well known, nodes of an acyclic network can be ordered so that for each arc (i, j) in the network, i < j. It is possible to determine this ordering, called a *topological ordering* of nodes, in O(m) time. Observe that pushes do not change the admissible network since they do not create new admissible arcs. The relabel operations, however, may create new admissible arcs and consequently may affect the topological ordering of nodes.

The wave algorithm examines each node in the topological order and if the node is active, then it performs a push/relabel step. When examined in this order, active nodes push flow to higher numbered nodes, which in turn push flow to even higher numbered nodes, and so on. A relabel operation changes the numbering of nodes and the topological order, and thus the method again starts to examine the nodes according to the topological order. However, if within n consecutive node examinations, the algorithm performs no relabel operation then all active nodes have discharged their excesses and the algorithm obtains a flow. Since the algorithm requires $O(n^2)$ relabel operations, we immediately obtain a bound of $O(n^3)$ on the number of node examinations. Each node examination entails at most one nonsaturating push. Consequently, the wave algorithm performs $O(n^3)$ nonsaturating pushes per Improve-Approximation.

We now describe a procedure for obtaining a topological order of nodes after each relabel operation. An initial topological ordering is determined using an O(m) algorithm. Suppose that while examining node i, the algorithm relabels it. Note that after the relabel operation at node i, the network contains no incoming admissible arc at node i (see the proof of Lemma 5.7). We then move node i from its present position in

the topological order to the first position. Notice that this altered ordering is a topological ordering of the new admissible network. This result follows from the facts (i) node i has no incoming admissible arc; (ii) for each outgoing admissible arc (i, j), node i precedes node j in the order; and (iii) the rest of the admissible network does not change and so the previous order is still valid. Thus the algorithm maintains an ordered set of nodes (possibly as a doubly linked list) and examines nodes in this order. Whenever it relabels a node i, the algorithm moves it to the first place in this order and again examines nodes in this order starting at node i.

We have established the following result.

Theorem 5.6. The cost scaling approach using the wave algorithm as a subroutine solves the minimum cost flow problem in $O(n^3 \log nC)$ time. ■

5.9. Double Scaling Algorithm

The double scaling approach combines ideas from both the RHS-scaling and cost scaling algorithms and obtains an improvement not obtained by either algorithm alone. For the sake of simplicity, we shall describe the double scaling algorithm on the uncapacitated transportation network $G = (N_1 \cup N_2, A)$, with N_1 and N_2 as the sets of supply and demand nodes respectively. A capacitated minimum cost flow problem can be solved by first transforming the problem into an uncapacitated transportation problem (as described in Section 2.4) and then applying the double scaling algorithm.

The double scaling algorithm is the same as the cost scaling algorithm discussed in the previous section except that it uses a more efficient version of the Improve-Approximation procedure. The Improve-Approximation procedure in the previous section relied on a "pseudoflow-push" method. A natural alternative would be to try an augmenting path based method. This approach would send flow from a node with excess to a node with deficit over an *admissible path*, i.e., a path in which each arc is admissible. A natural implementation of this approach would result in O(nm) augmentations since each augmentation would saturate at least one arc and, by Lemma 5.6, the algorithm requires O(nm) arc saturations. Thus, this approach does not seem to improve the $O(n^2m)$ bound of the generic Improve-Approximation procedure.

We can, however, use ideas from the RHS-scaling algorithm to reduce the number of augmentations to O(n log U) for an uncapacitated problem by ensuring that

each augmentation carries *sufficiently large* flow. This approach gives us an algorithm that does cost scaling in the outer loop and within each cost scaling phase performs a number of RHS-scaling phases; hence, this algorithm is called the *double scaling algorithm*. The advantage of the double scaling algorithm, contrasted with solving a shortest path problem in the RHS-scaling algorithm, is that the double scaling algorithm identifies an augmenting path in O(n) time on average over a sequence of n augmentations. In fact, the double scaling algorithm appears to be similar to the shortest augmenting path algorithm for the maximum flow problem; this algorithm, also requires O(n) time on average to find each augmenting path. The double scaling algorithm uses the following Improve-Approximation procedure.

```
procedure IMPROVE-APPROXIMATION-II(ε, x, π);
begin
    set x := 0 and compute node imbalances;
    π(j) := π(j) + ε , for all j ∈ N2;
    Δ := 2^⌈log U⌉;
    while the network contains an active node do
    begin
        S(Δ) := { i ∈ N1 ∪ N2 : e(i) ≥ Δ };
        while S(Δ) ≠ ø do
        begin (RHS-scaling phase)
            select a node k in S(Δ) and delete it from S(Δ);
            determine an admissible path P from node k to some node l
              with e(l) < 0;
            augment Δ units of flow on P and update x;
        end;
        Δ := Δ/2;
    end;
end;
```

We shall describe a method to determine admissible paths after first commenting on the correctness of this procedure. First, observe that $\bar{c}_{ij} \geq -\varepsilon$ for all $(i, j) \in A$ at the beginning of the procedure and, by adding ε to $\pi(j)$ for each $j \in N_2$, we obtain an $\varepsilon/2$-optimal (in fact, a 0-optimal) pseudoflow. The procedure always augments flow on admissible arcs and, from Lemma 5.4, this choice preserves the $\varepsilon/2$-optimality of the pseudoflow. Thus, at the termination of the procedure, we obtain an $\varepsilon/2$-optimal flow.

Further, as in the RHS-scaling algorithm, the procedure maintains the invariant property that all residual capacities are integer multiples of Δ and thus each augmentation can carry Δ units of flow.

The algorithm identifies an admissible path by gradually building the path. We maintain a *partial* admissible path P using a *predecessor index*, i.e., if $(u, v) \in P$ then *pred*(v) = u. At any point in the algorithm, we perform one of the following two steps, whichever is applicable, at the last node of P, say node i, terminating when the last node has a deficit.

advance(i). If the residual network contains an admissible arc (i, j), then add (i, j) to P. If $e(j) < 0$, then stop.

retreat(i). If the residual network does not contain an admissible arc (i, j), then update $\pi(i)$ to $\pi(i) + \varepsilon/2 + \min \{ \bar{c}_{ij} : (i, j) \in A(i) \text{ and } r_{ij} > 0\}$. If P has at least one arc, then delete (pred(i), i) from P.

The retreat step relabels (increases the potential of) node i for the purpose of creating new admissible arcs emanating from this node; in the process, the arc (pred(i), i) becomes inadmissible. Hence, we delete this arc from P. The proof of Lemma 5.4 implies that increasing the node potential maintains $\varepsilon/2$-optimality of the pseudoflow.

We next consider the complexity of this implementation of the Improve-Approximation procedure. Each execution of the procedure performs $1+\lceil \log U \rceil$ RHS-scaling phases. At the beginning of the Δ-scaling phase, $S(2\Delta) = \emptyset$, i.e., $\Delta \le e(i) < 2\Delta$ for each node $i \in S(\Delta)$. During the scaling phase, the algorithm augments Δ units of flow from a node k in $S(\Delta)$ to a node *l* with $e(l) < 0$. This operation reduces the excess at node k to a value less then Δ and ensures that the excess at node *l*, if there is any, is less than Δ. Consequently, each augmentation deletes a node from $S(\Delta)$ and after at most n augmentations, the method begins a new scaling phase. The algorithm thus performs a total of O(n log U) augmentations.

We next count the number of advance steps. Each advance step adds an arc to the partial admissible path, and a retreat step deletes an arc from the partial admissible path. Thus, there are two types of advance steps: (i) those that add arcs to an admissible path on which the algorithm later performs an augmentation; and (ii) those that are later cancelled by a retreat step. Since the set of admissible arcs is acyclic (by Lemma 5.7), after at most n advance steps of the first type, the algorithm will discover an admissible path

and will perform an augmentation. Since the algorithm requires a total of O(n log U) augmentations, the number of the first type of advance steps is at most $O(n^2 \log U)$. The algorithm performs advance steps at most $O(n^2)$ of the second type because each retreat step increases a node potential, and by Lemma 5.5, node potentials increase $O(n^2)$ times. The total number of advance steps, therefore, is $O(n^2 \log U)$.

The amount of time needed to identify admissible arcs is $O(\sum_{i=1}^{n} |A(i)| n) = O(nm)$ since between a potential increase of a node i, the algorithm will examine $|A(i)|$ arcs for testing admissibility. We have therefore established the following result.

Theorem 5.7. The double scaling algorithm solves the uncapacitated transportation problem in $O((nm + n^2 \log U) \log nC)$ time. ■

To solve the capacitated minimum cost flow problem ,we first transform it into an uncapacitated transportation problem and then apply the double scaling algorithm. We leave it as an exercise for the reader to show that how the transformation permits us to use the double scaling algorithm to solve the capacitated minimum cost flow problem O(nm log U log nC) time. The references describe further modest improvements of the algorithm. For problems that satisfy the similarity assumption, a variant of this algorithm using more sophisticated data structures is currently the fastest polynomial-time algorithm for most classes of the minimum cost flow problem.

5.10 Sensitivity Analysis

The purpose of sensitivity analysis is to determine changes in the optimum solution of a minimum cost flow problem resulting from changes in the data (supply/demand vector, capacity or cost of any arc). Traditionally, researchers and practitioners have conducted this sensitivity analysis using the primal simplex or dual simplex algorithms. There is, however, a conceptual drawback to this approach. The simplex based approach maintains a basis tree at every iteration and conducts sensitivity analysis by determining changes in the basis tree precipitated by changes in the data. The basis in the simplex algorithm is often degenerate, though, and consequently changes in the basis tree do not necessarily translate into the changes in the solution. Therefore, the simplex based approach does not give information about the changes in the solution as the data changes; instead, it tells us about the changes in the *basis tree*.

We present another approach for performing sensitivity analysis. This approach does not share the drawback we have just mentioned. For simplicity, we limit our discussion to a unit change of only a particular type. In a sense, however, this discussion is quite general: it is possible to reduce more complex changes to a sequence of the simple changes we consider. We show that the sensitivity analysis for the minimum cost flow problem essentially reduces to solving shortest path or maximum flow problems.

Let x^* denote an optimum solution of a minimum cost flow problem. Let π^* be the corresponding node potentials and $\bar{c}_{ij} = c_{ij} - \pi^*(i) + \pi^*(j)$ denote the reduced costs. Further, let $d(k, l)$ denote the shortest distance from node k to node l in the residual network with respect to the original arc lengths c_{ij}. Since for any directed path P from node k to node l, $\sum_{(i,j) \in P} \bar{c}_{ij} = \sum_{(i,j) \in P} c_{ij} - \pi(k) + \pi(l)$, $d(k, l)$ equals the shortest distance from node k to node l with respect to the arc lengths $\bar{c}_{ij}$ plus $(\pi^*(k) - \pi^*(l))$. At optimality, the reduced costs $\bar{c}_{ij}$ of all arcs in the residual network are nonnegative. Hence, we can compute $d(k, l)$ for all pairs of nodes k and l by solving n single-source shortest path problems with nonnegative arc lengths.

Supply/Demand Sensitivity Analysis

We first study the change in the supply/demand vector. Suppose that the supply/demand of a node k becomes $b(k) + 1$ and the supply/demand of another node l becomes $b(l) - 1$. (Recall from Section 1.1 that feasibility of the minimum cost flow problem dictates that $\sum_{i \in N} b(i) = 0$; hence, we must change the supply/demand values of two nodes by equal magnitudes, and must increase one value and decrease the other). Then x^* is a pseudoflow for the modified problem; moreover, this vector satisfies the dual feasibility conditions C5.6. Augmenting one unit of flow from node k to node l along the shortest path in the residual network $G(x^*)$ converts this pseudoflow into a flow. This augmentation changes the objective function value by $d(k, l)$ units. Lemma 5.1 implies that this flow is optimum for the modified minimum cost flow problem.

Arc Capacity Sensitivity Analysis

We next consider a change in an arc capacity. Suppose that the capacity of an arc (p, q) increases by one unit . The flow x^* is feasible for the modified problem. In

addition, if $\bar{c}_{pq} \geq 0$, it satisfies the optimality conditions C5.2 – C5.4; hence, it is an optimum flow for the modified problem. If $\bar{c}_{pq} < 0$, then condition C5.4 dictates that flow on the arc must equal its capacity. We satisfy this requirement by increasing the flow on the arc (p, q) by one unit, which produces a pseudoflow with an excess of one unit at node q and a deficit of one unit at node p. We convert the pseudoflow into a flow by augmenting one unit of flow from node q to node p along the shortest path in the residual network which changes the objective function value by an amount $c_{pq} + d(q, p)$. This flow is optimum from our observations concerning supply/demand sensitivity analysis.

When the capacity of the arc (p, q) decreases by one unit and flow on the arc is strictly less than its capacity, then x^* remains feasible, and hence optimun, for the modified problem. However, if the flow on the arc is at its capacity, we decrease the flow by one unit and augment one unit of flow from node p to node q along the shortest path in the residual network. This augmentation changes the objective function value by an amount $-c_{pq} + d(p, q)$.

The preceding discussion shows how to determine changes in the optimum solution value due to unit changes of any two supply/demand values or a unit change in any arc capacity by solving n single -source shortest path problems. We can, however, obtain useful upper bounds on these changes by solving only two shortest path problems. This observation uses the fact that $d(k, l) \leq d(k, 1) + d(1, l)$ for all pairs of nodes k and l. Consequently, we need to determine shortest path distances from node 1 to all other nodes, and from all other nodes to node 1 to compute upper bounds on all $d(k, l)$. Recent empirical studies have suggested that these upper bounds are very close to the actual values; often these upper bounds and the actual values are equal, and usually they are within 5% of each other.

Cost Sensitivity Analysis

Finally, we discuss changes in arc costs, which we assume are integral. Suppose that the cost of an arc (p, q) increases by one unit. This change increases the reduced cost of arc (p, q) by one unit as well. If $\bar{c}_{pq} = 1 < 0$ before the change, then after the change $\bar{c}'_{pq} \leq 0$. Similarly, if $\bar{c}_{pq} > 0$, before the change, then $\bar{c}'_{pq} \geq 0$ after the change. In both the cases, we preserve the optimality conditions. However, if $\bar{c}_{pq} = 0$ before the change and $x_{pq} > 0$, then after the change $\bar{c}_{pq} = 1 > 0$ and the solution violates the

condition C5.2. To satisfy the optimality condition of the arc, we must either reduce the flow on arc (p, q) to zero, or change the potentials so that the reduced cost of arc (p, q) becomes zero.

We first try to reroute the flow x^*_{pq} from node p to node q without violating any of the optimality conditions. We do so by solving a maximum flow problem defined as follows: (i) the flow on the arc (p, q) is set to zero, thus creating an excess of x^*_{pq} at node p and a deficit of x^*_{pq} at node q; (ii) define node p as the source node and node q as the sink node; and (iii) send a maximum of x^*_{pq} units from the source to the sink. We permit the maximum flow algorithm, however, to change flows only on arcs with zero reduced costs, since otherwise it would generate a solution that violates C5.2 and C5.4. Let v° denote the flow sent from node p to node q and x° denote the resulting arc flow. If $v^\circ = x^*_{pq}$, then x° denotes a minimum cost flow of the modified problem. In this case, the optimal objective function values of the original and modified problems are the same.

On the other hand, if $v^\circ < x^*_{pq}$ then the maximum flow algorithm yields an s-t cut (X, N– X) with the properties that $p \in X$, $q \in N - X$, and every forward arc in the cutset with zero reduced cost has others at the arc's capacitated. We then decrease the node potential of every node in $N - X$ by one unit. It is easy to verify by case analysis that this change in node potentials maintains the optimality conditions and, furthermore, decreases the reduced cost of arc (p, q) to zero. Consequently, we can set the flow on arc (p, q) equal to $x^*_{pq} - v^\circ$ and obtain a feasible minimum cost flow. In this case, the objective function value of the modified problem is $x^*_{pq} - v^\circ$ units more than that of the original problem.

5.11 Assignment Problem

The *assignment problem* is one of the best-known and most intensively studied special cases of the minimum cost network flow problem. As already indicated in Section 1.1 , this problem is defined by a set N_1, say of persons, a set N_2, say of objects ($|N_1| = |N_2| = n$) , a collection of node pairs $A \subseteq N_1 \times N_2$ representing possible person-to-object assignments, and a cost c_{ij} (possibly negative) associated with each element (i, j) in A. The objective is to assign each person to one object , choosing the assignment with

minimum possible cost. The problem can be formulated as the following linear program:

$$\text{Minimize} \quad \sum_{(i,j) \in A} c_{ij} x_{ij} \tag{5.18a}$$

subject to

$$\sum_{\{j : (i,j) \in A\}} x_{ij} = 1, \text{ for all } i \in N_1, \tag{5.18b}$$

$$\sum_{\{i : (i,j) \in A\}} x_{ij} = 1, \text{ for all } j \in N_2, \tag{5.18c}$$

$$x_{ij} \geq 0, \text{ for all } (i,j) \in A. \tag{5.18d}$$

The assignment problem is a minimum cost flow problem defined on a network G with node set $N = N_1 \cup N_2$, arc set A, arc costs c_{ij}, and supply /demand specified as $b(i) = 1$ if $i \in N_1$ and $b(i) = -1$ if $i \in N_2$. The network G has $2n$ nodes and $m = |A|$ arcs. The assignment problem is also known as the *bipartite matching problem.*

We use the following notation. A 0-1 solution x of (5.18) is an *assignment*. If $x_{ij} = 1$, then i is *assigned to* j and j is *assigned to* i. A 0-1 solution x satisfying $\sum_{\{j : (i,j) \in A\}} x_{ij} \leq 1$ for all $i \in N_1$ and $\sum_{\{i : (i,j) \in A\}} x_{ij} \leq 1$ for all $j \in N_2$ is called a *partial assignment*. Associated with any partial assignment x is an index set X defined as $X = \{(i,j) \in A : x_{ij} = 1\}$. A node not assigned to any other node is *unassigned*. .

Researchers have suggested numerous algorithms for solving the assignment problem. Several of these algorithms apply, either explicitly or implicitly, the successive shortest path algorithm for the minimum cost flow problem. These algorithms typically select the initial node potentials with the following values: $\pi(i) = 0$ for all $i \in N_1$ and $\pi(j) = \min \{c_{ij} : (i,j) \in A\}$ for all $j \in N_2$. All reduced costs defined by these node potentials are nonnegative. The successive shortest path algorithm solves the assignment problem as a sequence of n shortest path problems with nonnegative arc lengths, and consequently runs in $O(n\, S(n,m,C))$ time. (Note that S(n,m,C) is the time required to solve a shortest path problem with nonnegative arc lengths)

The relaxation approach is another popular approach, which is also closely related to the successive shortest path algorithm. The relaxation algorithm removes, or relaxes, the constraint (5.18c), thus allowing any object to be assigned to more than one person. This relaxed problem is easy to solve: assign each person i to an object j with the smallest c_{ij} value. As a result, some objects may be unassigned and other objects may be overassigned. The algorithm gradually builds a feasible assignment by identifying shortest paths from overassigned objects to unassigned objects and augmenting flows on these paths. The algorithm solves at most n shortest path problems. Because this approach always maintains the optimality conditions, it can solve the shortest path problems by implementations of Dijkstra's algorithm. Consequently, this algorithm also runs in $O(n\ S(n,m,C))$ time.

One well known solution procedure for the assignment problem, the *Hungarian method*, is essentially the primal-dual variant of the successive shortest path algorithm. The network simplex algorithm, with provisions for maintaining a strongly feasible basis, is another solution procedure for the assignment problem. This approach is fairly efficient in practice; moreover, some implementations of it provide polynomial time bounds. For problems that satisfy the similarity assumption, however, a cost scaling algorithm provides the best-known time bound for the assignment problem. Since these algorithms are special cases of other algorithms we have described earlier, we will not specify their details. Rather, in this section, we will discuss a different type of algorithm based upon the notion of an *auction*. Before doing so, we show another intimate connection between the assignment problem and the shortest path problem.

Assignments and Shortest Paths

We have seen that by solving a sequence of shortest path problems, we can solve any assignment problem. Interestingly, we can also use any algorithm for the assignment problem to solve the shortest path problem with arbitrary arc lengths. To do so, we apply the assignment algorithm twice. The first application determines if the network contains a negative cycle; and, if it doesn't, the second application identifies a shortest path. Both the applications use the node splitting transformation described in Section 2.4.

The node splitting transformation replaces each node i by two nodes i and i', replaces each arc (i, j) by an arc (i, j'), and adds an (artificial) zero cost arc (i, i'). We first note that the transformed network always has a feasible solution with cost zero:

namely, the assignment containing all artificial arcs (i, i'). We next show that the optimal value of the assignment problem is negative if and only if the original network has a negative cost cycle.

First, suppose the original network contains a negative cost cycle, j_1-j_2 -j_3 - ... j_k-j_1. Then the assigment $\{(j_1, j_2), (j_2, j_3), \ldots, (j_k, j_1), (j_{k+1}, j_{k+1}), \ldots, (j_n, j_n)\}$ has a negative cost. Therefore, the cost of the optimal assignment must be negative. Conversely, suppose the cost of an optimal assignment is negative. This solution must contain at least one arc of the form (i, j') with $i \neq j$. Consequently, the assignment must contain a set of arcs of the form $PA = \{(j_1, j_2'), (j_2, j_3'), \ldots, (j_k, j_1')\}$. The cost of this "partial" assignment is nonpositive, because it can be no more expensive than the partial assignment $\{(j_1, j_1'), (j_2, j_2'), \ldots, (j_k, j_k')\}$. Since the optimal assignment cost is negative, some partial assignment PA must be negative. But then by construction of the transformed network, the cycle $j_1 - j_2 - \ldots - j_k - j_1$ is a negative cost cycle in the original network.

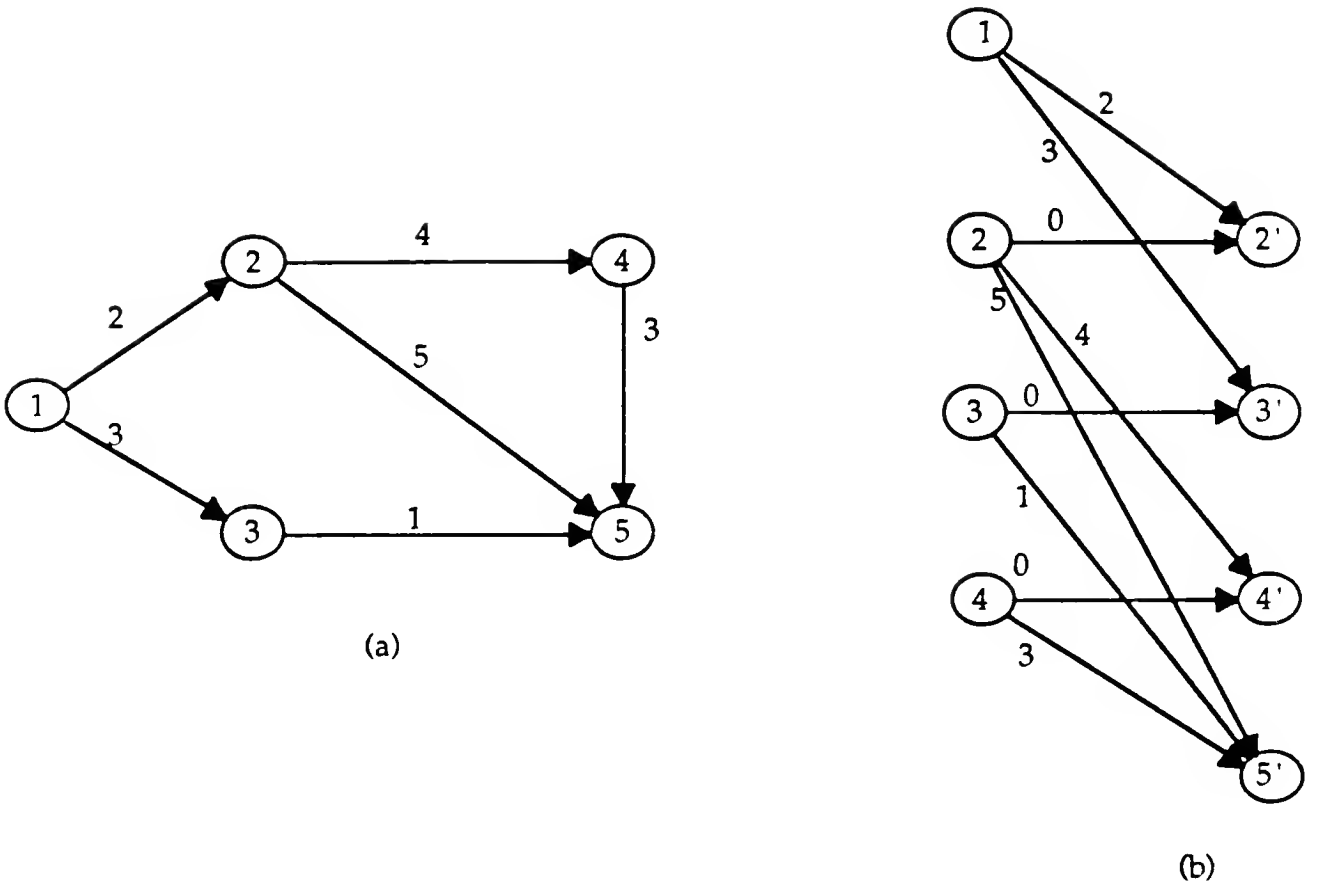


Figure 5.3. (a) The original network. (b) The transformed network.

If the original network contains no negative cost cycle, then we can obtain a shortest path between a specific pair of nodes, say from node 1 to node n, as follows. We consider the transformed network as described earlier and delete the nodes 1' and n and the arcs incident to these nodes. See Figure 5.3 for an example of this transformation. Now observe that each path from node 1 to node n in the original network has a corresponding assignment of the same cost in the transformed network, and the converse is also true. For example, the path 1-2-5 in Figure 5.3(a) has the corresponding assignment {(1, 2'), (2, 5'), (3, 3'), (4, 4')} in Figure 5.3(b), and an assignment {(1, 2'), (2, 4'), (4, 5'), (3, 3')} in Figure 5.3(b) has the corresponding path 1-2-4-5 in Figure 5.3(a). Consequently, an optimum assignment in the transformed network gives a shortest path in the original network.

The Auction Algorithm

We now describe an algorithm for the assignment problem known as the *auction algorithm*. We first describe a pseudopolynomial time version of the algorithm and then incorporate scaling to make the algorithm polynomial time. This scaling algorithm is an instance of the bit-scaling algorithm described in Section 1.6. To describe the auction algorithm, we consider the maximization version of the assignment problem, since this version appears more natural for interpreting the algorithm.

Suppose n persons want to buy n cars that are to be sold by auction. Each person i is interested in a subset of cars, and has a nonnegative utility u_{ij} for car j for each $(i, j) \in A(i)$. The objective is to find an assignment with maximum utility. We can set $c_{ij} = -u_{ij}$ to reduce this problem to (5.18). Let $C = \max \{|u_{ij}| : (i, j) \in A\}$. At each stage of the algorithm, there is an asking price for car j, represented by *price(j)*. For a given set of asking prices, the *marginal utility* of person i for buying car j is u_{ij} - price(j). At each iteration, an unassigned person bids on a car that has the highest marginal utility. We assume that all utilities and prices are measured in dollars.

We associate with each person i a number *value(i)*, which is an upper bound on that person's highest marginal utility, i.e., value(i) $\geq \max \{u_{ij} - \text{price}(j) : (i, j) \in A(i)\}$. We call a bid (i, j) *admissible* if value(i) = u_{ij} - price(j) and *inadmissible* otherwise. The algorithm requires every bid in the auction to be admissible. If person i is next in turn to bid and has no admissible bid, then value(i) is too high and we decrease this value to $\max \{u_{ij} - \text{price}(j) : (i, j) \in A(i)\}$.

So the algorithm proceeds by persons bidding on cars. If a person i makes a bid on car j, then the price of car j goes up by \$1; therefore, subsequent bids are of higher value. Also, person i is assigned to car j. The person k who was the previous bidder for car j, if there was one, becomes unassigned. Subsequently, person k must bid on another car. As the auction proceeds, the prices of cars increase and hence the marginal values to the persons decrease. The auction stops when each person is assigned a car. We now describe this bidding procedure algorithmically. The procedure starts with some valid choices for value(i) and price(j). For example, we can set price(j) = 0 for each car j and value(i) = max $\{u_{ij} : (i, j) \in A(i)\}$ for each person i. Although this initialization is sufficient for the pseudopolynomial time version, the polynomial time version requires a more clever initialization. At termination, the procedure yields an *almost* optimum assignment x°.

```
procedure BIDDING(u, x°, value, price);
begin
    let the initial assignment be a null assignment;
    while some person is unassigned do
    begin
        select an unassigned person i;
        if some bid (i, j) is admissible then
        begin
            assign person i to car j;
            price(j) : = price(j) + 1;
            if person k was already assigned to car j, then
                person k becomes unassigned;
        end
        else update value(i) : = max {u_ij - price(j) : (i, j) ∈ A(i)};
    end;
    let x° be the current assignment;
end;
```

We now show that this procedure gives an assignment whose utility is within \$n of the optimum utility. Let x° denote a partial assignment at some point during the execution of the auction algorithm and x^* denote an optimum assignment. Recall that value(i) is always an upper bound on the highest marginal utility of person i, i.e., value(i) $\geq u_{ij}$ - price(j) for all $(i, j) \in A(i)$. Consequently,

$$\sum_{(i,j) \in X^*} u_{ij} \leq \sum_{i \in N_1} \text{value}(i) + \sum_{j \in N_2} \text{price}(j) \qquad (5.19)$$

The partial assignment x° also satisfies the condition

$$\text{value}(i) = u_{ij} - \text{price}(j) + 1, \text{ for all } (i, j) \in X^\circ, \qquad (5.20)$$

because at the time of bidding value(i) = u_{ij} - price(j) and immediately after the bid, price(j) goes up by \$1. Let $UB(x^\circ)$ be defined as follows.

$$UB(x^\circ) = \sum_{(i,j) \in X^\circ} u_{ij} + \sum_{i \in N_1^\circ} \text{value}(i), \qquad (5.21)$$

with N_1° denoting the unassigned persons in N_1. Using (5.20) in (5.21) and observing that unassigned cars in N_2 have zero prices, we obtain

$$UB(x^\circ) \geq \sum_{i \in N_1} \text{value}(i) + \sum_{j \in N_2} \text{price}(j) - n. \qquad (5.22)$$

Combining (5.19) and (5.22) yields

$$UB(x^\circ) \geq -n + \sum_{(i,j) \in X^*} u_{ij}. \qquad (5.23)$$

As we show in our discussion to follow, the algorithm can change the node values and prices at most a finite number of times. Since the algorithm will either modify a node value or node price whenever x° is not an assignment, within a finite number of steps the method must terminate with a complete assignment x°. Then $UB(x^\circ)$ represents the utility of this assignment (since N_1^0 is empty). Hence, the utility of the assignment x° is at most \$n less than the maximum utility.

It is easy to modify the method, however, to obtain an optimum assignment. Suppose we multiply all utilities u_{ij} by (n+1) before applying the Bidding procedure. Since all utilities are now multiples of (n+1), two assignments with distinct total utility will differ by at least (n+1) units. The procedure yields an assignment that is within n units of the optimum value and, hence, must be optimal.

We next discuss the complexity of the Bidding procedure as applied to the assignment problem with all utilities multiplied by (n+1). In this modified problem, the largest utility is C' = (n+1)C. We first show that the value of any person decreases O(nC')

times. Since all utilities are nonnegative, (5.23) implies $UB(x^\circ) \geq -n$. Substituting this inequality in (5.21) yields

$$\sum_{i \in N_1^\circ} \text{value}(i) \geq -n(C' + 1).$$

Since value(i) decreases by at least one unit each time it changes, this inequality shows that the value of any person decreases at most O(nC') times. Since decreasing the value of a person i once takes $O(|A(i)|)$ time, the total time needed to update values of all persons is $O\left(\sum_{i \in N_1} n\,|A(i)|\,C'\right) = O(nmC')$.

We next examine the number of iterations performed by the procedure. Each iteration either decreases the value of a person i or assigns the person to some car j. By our previous arguments, the values change $O(n^2C')$ times in total. Further, since value(i) > u_{ij} – price(j) after person i has been assigned to car j and the price of car j increases by one unit, a person i can be assigned at most $|A(i)|$ times between two consecutive decreases in value(i). This observation gives us a bound of O(nmC') on the total number of times all bidders become assigned. As can be shown, using the "current arc" data structure permits us to locate admissible bids in O(nmC') time. Since C' = nC, we have established the following result.

Theorem 5.8. The auction algorithm solves the assignment problem in $O(n^2mC)$ time. ■

The auction algorithm is potentially very slow because it can increase prices (and thus decreases values) in small increments of \$1 and the final prices can be as large as n^2C (the values as small as $-n^2C$). Using a scaling technique in the auction algorithm ensures that the prices and values do not change *too many* times. As in the *bit -scaling* technique described in Section 1.6, we decompose the original problem into a sequence of O(log nC) assignment problems and solve each problem by the auction algorithm. We use the optimum prices and values of a problem as a starting solution of the subsequent problem and show that the prices and values change only O(n) times per scaling phase. Thus, we solve each problem in O(nm) time and solve the original problem in O(nm log nC) time.

The scaling version of the auction algorithm first multiplies all utilities by (n+1) and then solves a sequence of $K = \lceil \log (n+1)C \rceil$ assignment problems $P_1, P_2, \ldots,$

P_K. The problem P_k is an assignment problem in which the utility of arc (i,j) is the k leading bits in the binary representation of u_{ij}, assuming (by adding leading zeros if necessary) that each u_{ij} is K bits long. In other words, the problem P_k has the arc utilities $u_{ij}^k = \lfloor u_{ij} / 2^{K-k} \rfloor$. Note that in the problem P_1, all utilities are 0 or 1, and subsequently $u_{ij}^{k+1} = 2u_{ij}^k + \{0 \text{ or } 1\}$, depending upon whether the newly added bit is 0 or 1. The scaling algorithm works as follows:

```
algorithm ASSIGNMENT;
begin
    multiply all u_ij by (n+1);
    K : = ⌈log (n+1)C⌉;
    price(j) : = 0 for each car j;
    value(i) : = 0 for each person i;
    for k : = 1 to K do
    begin
        let u_ij^k : = ⌊ u_ij / 2^(K-k) ⌋ for each (i, j) ∈ A;
        price(j) : = 2 price(j) for each car j;
        value(i) : = 2 value (i) + 1 for each person i;
        BIDDING(u^k, x°, value, price);
    end;
end;
```

The assignment algorithm performs a number of cost scaling phases. In the k-th scaling phase, it obtains a near-optimum solution of the problem with the utilities u_{ij}^k. It is easy to verify that before the algorithm invokes the Bidding procedure, prices and values satisfy value(i) $\geq$ max $\{u_{ij}$ - price(j) : (i, j) $\in$ A(i)$\}$, for each person i. The Bidding procedure maintains these conditions throughout its execution. In the last scaling phase, the algorithm solves the assignment problem with the original utilities and obtains an optimum solution of the original problem. Observe that in each scaling phase, the algorithm starts with a null assignment; the purpose of each scaling phase is to obtain good prices and values for the subsequent scaling phase.

We next discuss the complexity of this assignment algorithm. The crucial result is that the prices and values change only O(n) times during each execution of the

Bidding procedure. We define the reduced utility of an arc (i, j) in the k-th scaling phase as

$$\bar{u}_{ij} = u^k_{ij} - \text{price}(j) - \text{value}(i).$$

In this expression, price(j) and value(i) have the values computed just before calling the Bidding procedure. For any assignment x, we have

$$\sum_{(i,j) \in X} \bar{u}_{ij} = \sum_{(i,j) \in X} u^k_{ij} - \sum_{j \in N_2} \text{price}(j) - \sum_{i \in N_1} \text{value}(i).$$

Consequently, for a given set of prices and values, the reduced utility of an assignment differs from the utility of that assignment by a constant amount. Therefore, an assignment that maximizes the reduced utility also maximizes the utility. Since $\text{value}(i) \geq u^k_{ij} - \text{price}(j)$ for each $(i, j) \in A$, we have

$$\bar{u}_{ij} \leq 0, \text{ for all } (i, j) \in A. \tag{5.24}$$

Now consider the reduced utilities of arcs in the assignment x^{k-1} (the final assignment at tl.e end of the (k-1)-st scaling phase). The equality (5.20) implies that

$$u^{k-1}_{ij} - \text{price}'(j) - \text{value}'(i) = -1, \text{ for all } (i, j) \in x^{k-1}, \tag{5.25}$$

where price'(j) and value'(i) are the corresponding values at the end of the (k-1)-st scaling phase. Before calling the Bidding procedure, we set price(j) = 2 price'(j), value(i) = 2 value'(i) + 1, and $u^k_{ij} = 2\,u^{k-1}_{ij}$ + {0 or 1}. Substituting these relationships in (5.25), we find that the reduced utilities $\bar{u}_{ij}$ of arcs in x^{k-1} are either -2 or -3. Hence, the optimum reduced utility is at least -3n. If x° is some partial assignment in the k-th scaling phase, then (5.23) implies that $UB(x^\circ) \geq -4n$. Using this result and (5.24) in (5.21) yields

$$\sum_{i \in N_1^\circ} \text{value}(i) \geq -4n. \tag{5.26}$$

Hence, for any i, value(i) decreases O(n) times. Using this result in the proof of Theorem 5.7, we observe that the Bidding procedure would terminate in O(nm) time. The assignment algorithm applies the Bidding procedure O(log nC) times and, consequently, runs in O(nm log nC) time. We summarize our discussion.

Theorem 5.9. The scaling version of the auction algorithm solves the assignment problem in O(nm log nC) time. ■

The scaling version of the auction algorithm can be further improved to run in $O(\sqrt{n}\, m \log nC)$ time. This improvement is based on the following implication of (5.26). If we prohibit person i from bidding if value(i) $\geq 4\sqrt{n}$, then by (5.26) the number of unassigned persons is at most $\sqrt{n}$. Hence, the algorithm takes $O(\sqrt{n}\, m)$ time to assign $n - \lceil \sqrt{n} \rceil$ persons and $O((n - \lceil \sqrt{n} \rceil)m)$ time to assign the remaining $\lceil \sqrt{n} \rceil$ persons. For example, if n = 10,000, then the auction algorithm would assign the first 99% of the persons in 1% of the overall running time and would assign the remaining 1% of the persons in the remaining 99% of the time. We therefore terminate the execution of the auction algorithm when it has assigned all but $\lceil \sqrt{n} \rceil$ persons and use successive shortest path algorithms to assign these persons. It so happens that the shortest paths have length O(n) and thus Dial's algorithm, as described in Section 3.2, will find these shortest paths in O(m) time. This version of the auction algorithm solves a scaling phase in $O(\sqrt{n}\, m)$ time and its overall running time is $O(\sqrt{n}\, m \log nC)$. If we invoke the similarity assumption, then this version of the algorithm currently has the best known time bound for solving the assignment problem .

6. Reference Notes

In this section, we present reference notes on topics covered in the text. This discussion has three objectives: (i) to review important theoretical contributions on each topic, (ii) to point out inter-relationships among different algorithms, and (iii) to comment on the empirical aspects of the algorithms.

6.1 Introduction

The study of network flow models predates the development of linear programming techniques. The first studies in this problem domain, conducted by Kantorovich [1939], Hitchcock [1941], and Koopmans [1947], considered the transportation problem, a special case of the minimum cost flow problem. These studies provided some insight into the problem structure and yielded incomplete algorithms. Interest in network problems grew with the advent of the simplex algorithm by Dantzig in 1947. Dantzig [1951] specialized the simplex algorithm for the tranportation problem. He noted the traingularity of the basis and integrality of the optimum solution. Orden [1956] generalized this work by specializing the simplex algorithm for the uncapacitated minimum cost flow problem. The network simplex algorithm for the capacitated minimum cost flow problem followed from the development of the bounded variable simplex method for linear programming by Dantzig [1955]. The book by Dantzig [1962] contains a thorough description of these contributions along with historical perspectives.

During the 1950's, researchers began to exhibit increasing interest in the minimum cost flow problem as well as its special cases--the shortest path problem, the maximum flow problem and the assignment problem -- mainly because of their important applications. Soon researchers developed special purpose algorithms to solve these problems. Dantzig, Ford and Fulkerson pioneered those efforts. Whereas Dantzig focused on the primal simplex based algorithms, Ford and Fulkerson developed primal-dual type combinatorial algorithms to solve these problems. Their book, Ford and Fulkerson [1962], presents a thorough discussion of the early research conducted by them and by others. It also covers the development of flow decomposition theory, which is credited to Ford and Fulkerson.

Since these pioneering works, network flow problems and their generalizations emerged as major research topics in operations research; this research

is documented in thousands of papers and many text and reference books. We shall be surveying many important research papers in the following sections. Several important books summarize developments in the field and serve as a guide to the literature: Ford and Fulkerson [1962] (**Flows in Networks**), Berge and Ghouila-Houri [1962] (**Programming , Games and Transportation Networks**), Iri [1969] (**Network Flows, Transportation and Scheduling**), Hu [1969] (**Integer Programming and Network Flows**), Frank and Frisch [1971] (**Communication, Transmission and Transportation Networks**), Potts and Oliver [1972] (**Flows in Transportation Networks**), Christophides [1975] (**Graph Theory: An Algorithmic Approach**), Murty [1976] (**Linear and Combinatorial Programming**), Lawler [1976] (**Combinatorial Optimization: Networks and Matroids**), Bazaraa and Jarvis [1978] (**Linear Programming and Network Flows**), Minieka [1978] (**Optimization Algorithms for Networks and Graphs**), Kennington and Helgason [1980] (**Algorithms for Network Programming**), Jensen and Barnes [1980] (**Network Flow Programming**), Phillips and Garcia-Diaz [1981] (**Fundamentals of Network Analysis**), Swamy and Thulsiraman [1981] (**Graphs, Networks and Algorithms**), Papadimitriou and Steiglitz [1982] (**Combinatorial Optimization: Algorithms and Complexity**), Smith [1982] (**Network Optimization Practice**), Syslo, Deo, and Kowalik [1983] (**Discrete Optimization Algorithms**), Tarjan [1983] (**Data Structures and Network Algorithms**), Gondran and Minoux [1984] (**Graphs and Algorithms**), Rockafellar [1984] (**Network Flows and Monotropic Optimization**), and Derigs [1988] (**Programming in Netorks and Graphs**). As an additional source of references, the reader might consult the bibliography on network optimization prepared by Golden and Magnanti [1977] and the extensive set of references on integer programming compiled by researchers at the University of Bonn (Kastning [1976], Hausman [1978], and Von Randow [1982, 1985]).

Since the applications of network flow modelsa are so pervasive, no single source provides a comprehensive account of network flow models and their impact on practice. Several researchers have prepared general surveys of selected application areas. Notable among these is the paper by Glover and Klingman [1976] on the applications of minimum cost flow and generalized minimum cost flow problems. A number of books written in special problem domains also contain valuable insight about the range of applications of network flow models. Examples in this category are the paper by Bodin, Golden, Assad and Ball [1983] on vehicle routing and scheduling problems, books on communication networks by Bertsekas

and Gallager [1987] and on transportation planning by Sheffi [1985], as well as a collection of survey articles on facility location edited by Francis and Mirchandani [1988]. Golden [1988] has described the census rounding application given in Section 1.1.

General references on data structure serve as a useful backdrop for the algorithms presented in this chapter. The book by Aho, Hopcroft and Ullman [1974] is an excellent reference for simple data structures as arrays, linked lists, doubly linked lists, queues, stacks, binary heaps or d-heaps. The book by Tarjan [1983] is another useful source of references for these topics as well as for more complex data structures such as dynamic trees.

We have mentioned the "similarity assumption" throughout the chapter. Gabow [1985] coined this term in his paper on scaling algorithm for combinatorial optimization problems. This important paper, which contains scaling algorithms for several network problems, greatly helped in popularizing scaling techniques.

6.2 Shortest Path Problem

The shortest path problem and its generalizations have a voluminous research literature. As a guide to these results, we refer the reader to the extensive bibliographies compiled by Gallo, Pallattino, Ruggen and Starchi [1982] and Deo and Pang [1984]. This section, which summarizes some of this literature, focuses especially on issues of computational complexity.

Label Setting Algorithms

The first label setting algorithm was suggested by Dijkstra [1959], and independently by Dantzig [1960] and Whiting and Hillier [1960]. The original implementation of Dijkstra's algorithm runs in $O(n^2)$ time which is the optimal running time for fully dense networks (those with $m = \Omega(n^2)$), since any algorithm must examine every arc. However, improved running times are possible for sparse networks. The following table summarizes various implementations of Dijkstra's algorithm that have been designed to improve the running time in the worst case or in practice. In the table, $d = [2 + m/n]$ represents the average degree of a node in the network plus 2.

#	Discoverers	Running Time
1	Dijkstra [1959]	$O(n^2)$
2	Williams [1964]	O(m log n)
3	Dial [1969], Wagner[1976]	O(m + n C)
4	Johnson [1977a]	$O(m \log_d n)$
5	Johnson [1977b]	O((m + n log C)log log C)
6	Boas, Kaas and Zijlstra [1977]	O(nC + m log log nC)
7	Denardo and Fox [1979]	O(m log logC + n log C)
8	Johnson [1982]	O(m log log C)
9	Fredman and Tarjan [1984]	O(m + n log n)
10	Gabow [1985]	$O(m \log_d C)$
11	Ahuja, Mehlhorn, Orlin and Tarjan [1988]	(a) O(m + n log C) (b) $O\left(m + \frac{n \log C}{\log \log C}\right)$ (c) $O(m + n \sqrt{\log C})$

Table 6.1. Running times of label setting shortest path algorithms.

Computer scientists have tried to improve the worst-case complexity of Dijkstra's algorithm by using improved data structures. When implemented with a *binary heap* data structure, the algorithm takes O(log n) time for each node selection (and the subsequent deletion) step and each distance update; consequently, this implementation of Dijkstra's algorithm runs in O(m log n) time. For sparse networks, this time is better than $O(n^2)$, but it is worse for dense networks. The *d-heap* data structure suggested by Johnson [1977a] takes $O(d \log_d n)$ time for each node selection (and the subsequent deletion) step and $O(\log_d n)$ for each distance update. For $d = 2 + m/n$, this approach leads to a time bound of $O(m \log_d n)$. For very sparse networks (those with $m = O(n)$), this time reduces to O(n log n) and for very dense networks (those with $m = \Omega(n^2)$), it becomes $O(n^2)$. For all other ranges of densities as well, the running time of this algorithm is better than that of either the original implementation or the binary heap implementation of Dijkstra's algorithm.

For problems that satisfy the similarity assumption, Gabow's [1985] scaling algorithm achieves the same time bound. Gabow decomposed the original problem into $\lceil \log_d C \rceil$ scaled problems and solved each scaled problem in O(m) time by Dial's algorithm, thus yielding an $O(m \log_d C)$ algorithm for the shortest path problem.

Boas, Kaas and Zijlstra [1977] suggested a data structure whose analysis depends upon the largest key D stored in a heap. The initialization of this algorithm takes O(D) time and each heap operation takes O(log log D). When Dijkstra's algorithm is implemented using this data structure, it runs in O(nC + m log log nC) time. Johnson [1982] suggested an improvement of this data structure and used it to implement Dijkstra's algorithm in O(m log log C) time.

The best strongly polynomial-time algorithm to date is due to Fredman and Tarjan [1984] who use a *Fibonacci heap* data structure. The Fibonacci heap is an ingenious, but somewhat complex, data structure that takes an average of O(log n) time for each node selection (and the subsequent deletion) step and an average of O(1) time for each distance update. Consequently, this data structure implements Dijkstra's algorithm in O(m + n log n) time.

Dial [1969] suggested his implementation of Dijkstra's algorithm because of its encouraging empirical performance. This algorithm was independently discovered by Wagner[1976]. Dial, Glover, Karney and Klingman [1979] have proposed an improved version of Dial's algorithm, which runs better in practice. Though Dial's algorithm is only pseudopolynomial-time, its successors have had improved worst-case behavior. Denardo and Fox [1979] suggest several such improvements. Observe that if $w = \max [1, \min\{c_{ij}: (i,j) \in A\}]$, then we can use buckets of width w in Dial's algorithm, hence reducing the number of buckets from 1+ C to 1+(C/w). The correctness of this observation follows from the fact that if d^* is the current minimum temporary distance labels, then the algorithm will modify no other temporary distance label in the range $[d^*, d^* + w - 1]$ since each arc has length at least w - 1. Then, using a multiple level bucket scheme, Denardo and Fox implemented the shortest path algorithm in $O(\max\{k\, C^{1/k}, m \log (k+1), nk(1+C^{1/k}/w)\}]$ time for any choice of k. Choosing k = log C yields a time bound of O(m log log C + n log C). Depending on n,m and C, other choices might lead to a modestly better time bound.

Johnson [1977b] proposed a related bucket scheme with exponentially growing widths and obtained the running time of O((m+n log C)log log C). This data structure is the same as the R-heap data structure described in Section 3.3, except that it performs binary search over O(log C) buckets to insert nodes into buckets during the redistribution of ranges and the distance updates. The R-heap implementation replaces the binary search by a sequential search and improves the running time by a

factor of O(log log C). Ahuja, Mehlhorn, Orlin and Tarjan [1988] suggested the R-heap implementation and its further improvements, as described next.

The R-heap implementation described in section 3.3 uses a single level bucket system. A two-level bucket system improves further on the R-heap implementation of Dijkstra's algorithm. The two-level data structure consists of K (big) buckets, each bucket being further subdivided into L (small) *subbuckets*. During redistribution, the two-level bucket system redistributes the range of a subbucket over all of its previous buckets. This approach permits the selection of much larger width of buckets, thus reducing the number of buckets. By using $K = L = 2 \log C/\log \log C$, this two-level bucket system version of Dijkstra's algorithm runs in $O(m+n \log C/\log \log C)$ time. Incorporating a generalization of the Fibonacci heap data structure in the two-level bucket system with appropriate choices of K and L further reduces the time bound to $O(m + n\sqrt{\log C})$. If we invoke the similarity assumption, this approach currently gives the fastest worst-case implementation of Dijkstra's algorithm for all classes of graphs except very sparse ones, for which the algorithm of Johnson [1982] appears more attractive. The Fibonacci heap version of two-level R-heap is very complex, however, and so it is unlikely that this algorithm would perform well in practice.

Label Correcting Algorithm

Ford [1956] suggested, in skeleton form, the first label correcting algorithm for the shortest path problem. Subsequently, several other researchers - Ford and Fulkerson [1962] and Moore [1957] - studied the theoretical properties of the algorithm. Bellman's [1958] algorithm can also be regarded as a label correcting algorithm. Though specific implementations of label correcting algorithms run in O(nm) time, the most general form is nonpolynomial-time, as shown by Edmonds [1970].

Researchers have exploited the flexibility inherent in the generic label correcting algorithm to obtain algorithms that are very efficient in practice. The modification that adds a node to the LIST (see the description of the Modified Label Correcting Algorithm given in Section 3.4.) at the front if the algorithm has previously examined the node earlier and at the end otherwise, is probably the most popular. This modification was conveyed to Pollack and Wiebenson [1960] by D'Esopo, and later refined and tested by Pape [1974]. We shall subsequently refer to this algorithm as D'Esopo and Pape's algorithm. A FORTRAN listing of this

algorithm can be found in Pape [1980]. Though this modified label correcting algorithm has excellent computational behavior in the worst-case it runs in exponential time, as shown by Kershenbaum [1981].

Glover, Klingman and Phillips [1985] proposed a generalization of the FIFO label correcting algorithm, called the *partitioning shortest path (PSP) algorithm*. For general networks, the PSP algorithm runs in O(nm) time, while for networks with nonnegative arc lengths it runs in $O(n^2)$ time and has excellent computational behavior. Other variants of the label correcting algorithms and their computational attributes can be found in Glover, Klingman, Phillips and Schneider [1985].

Researchers have been interested in developing polynomial-time primal simplex algorithms for the shortest path problem. Dial, Glover, Karney and Klingman [1979] and Zadeh [1979] showed that Dantzig's pivot rule (i.e., pivoting in the arc with largest violation of optimality condition) for the shortest path problem starting from an artificial basis leads to Dijkstra's algorithm. Thus, the number of pivots is O(n) if all arc costs are nonnegative. Primal simplex algorithms for the shortest path problem with arbitrary arc lengths are not that efficient. Akgul [1985a] developed a simplex algorithm for the shortest path problem that performs $O(n^2)$ pivots. Using simple data structures, Akgul's algorithm runs in $O(n^3)$ time which can be reduced to $O(nm + n^2 \log n)$ using the Fibonacci heap data structure. Goldfarb, Hao and Kai [1986] described another simplex algorithm for the shortest path problem: the number of pivots and running times for this algorithm are comparable to those of Akgul's algorithm. Orlin [1985] showed that the simplex algorithm with Dantzig's pivot rule solves the shortest path problem in $O(n^2 \log nC)$ pivots. Ahuja and Orlin [1988] recently discovered a scaling variation of this approach that performs $O(n^2 \log C)$ pivots and runs in $O(nm \log C)$ time. This algorithm uses simple data structures, uses very natural pricing strategies, and also permits partial pricing .

All Pair Shortest Path Algorithms

Most algorithms that solve the all pair shortest path problem involve matrix manipulation. The first such algorithm appears to be a part of the folklore. Lawler [1976] describes this algorithm in his textbook. The complexity of this algorithm is $O(n^3 \log n)$, which can be improved slightly by using more sophisticated matrix multiplication procedures. The algorithm we have presented is due to Floyd [1962] and is based on a theorem by Warshall [1962]. This algorithm runs in $O(n^3)$ time and

is also capable of detecting the presence of negative cycles. Dantzig [1967] devised another procedure requiring exactly the same order of calculations. The bibliography by Deo and Pang [1984] contains references for several other all pair shortest path algorithms.

From a worst-case complexity point of view, however, it might be desirable to solve the all pair shortest path problem as a sequence of single source shortest path problems. As pointed out in the text, this approach takes O(nm) time to construct an equivalent problem with nonnegative arc lengths and takes O(n S(n,m,C)) time to solve the n shortest path problems (recall that S(n,m,C) is the time neede to solve a shortest path problem with nonnegative arc lengths). For very dense networks, the algorithm by Fredman [1976] is faster than this approach in the worst-case complexity.

Computational Results

Researchers have extensively tested shortest path algorithms on a variety of network classes. The studies due to Gilsinn and Witzgall [1973], Pape [1974], Kelton and Law [1978], Van Vliet [1978], Dial, Glover, Karney and Klingman [1979], Denardo and Fox [1979], Imai and Iri [1984], Glover, Klingman, Phillips and Schneider [1985] , and Gallo and Pallottino [1988] are representative of these contributions.

Unlike the worst-case results, the computational performance of an algorithm depends upon many factors: for example, the manner in which the program is written; the language, compiler and the computer used; and the distribution of networks on which the algorithm is tested. Hence, the results of computational studies are only suggestive, rather than conclusive. The results of these studies also depend greatly upon the density of the network. These studies generally suggest that Dial's algorithm is the best label setting algorithm for the shortest path problem. It is faster than the original $O(n^{2})$ implementation, the binary heap, d-heap or the Fibonacci heap implementation of Dijkstra's algorithm for all network classes tested by these researchers. Denardo and Fox [1979] also find that Dial's algorithm is faster than their two-level bucket implementation for all of their test problems; however, extrapolating the results, they observe that their implementation would be faster for very large shortest path problems. Researchers have not yet tested the R-heap implementation and so at this moment no comparison with Dial's algorithm is available.

Among the label correcting algorithms, the algorithms by D'Esopo and Pape and by Glover, Klingman, Phillips and Schneider [1985] are the two fastest. The study by Glover et al. finds that their algorithm is superior to D'Esopo and Pape's algorithm. Other researchers have also compared label setting algorithms with label correcting algorithms. Studies generally suggest that, for very dense networks, label setting algorithms are superior and, for sparse networks, label correcting algorithms perform better.

Kelton and Law [1978] have conducted a computational study of several all pair shortest path algorithms. This study indicates that Dantzig's [1967] algorithm with a modification due to Tabourier [1973] is faster (up to two times) than the Floyd-Warshall algorithm described in Section 3.5. This study also finds that matrix manipulation algorithms are faster than a successive application of a single-source shortest path algorithm for very dense networks, but slower for sparse networks.

6.3 Maximum Flow Problem

The maximum flow problem is distinguished by the long succession of research contributions that have improved upon the worst-case complexity of algorithms; some, but not all, of these improvements have produced improvements in practice.

Several researchers - Dantzig and Fulkerson [1956], Ford and Fulkerson [1956] and Elias, Feinstein and Shannon [1956] - independently established the max-flow min-cut theorem. Fulkerson and Dantzig [1955] solved the maximum flow problem by specializing the primal simplex algorithm, whereas Ford and Fulkerson [1956] and Elias et al. [1956] solved it by augmenting path algorithms. Since then, researchers have developed a number of algorithms for this problem; Figure 6.2 summarizes the running times of some of these algorithms. In the figure, n is the number of nodes, m is the number of arcs, and U is an upper bound on the integral arc capacities. The algorithms whose time bounds involve U assume integral capacities; the bounds specified for the other algorithms apply to problems with arbitrary rational or real capacities.

#	Discoverers	Running Time
1	Edmonds and Karp [1972]	$O(nm^2)$
2	Dinic [1970]	$O(n^2m)$
3	Karzanov [1974]	$O(n^3)$
4	Cherkasky [1977]	$O(n^2\sqrt{m})$
5	Malhotra, Kumar and Maheshwari [1978]	$O(n^3)$
6	Galil [1980]	$O(n^{5/3}m^{2/3})$
7	Galil and Naamad [1980]; Shiloach [1978]	$O(nm \log^2 n)$
8	Shiloach and Vishkin [1982]	$O(n^3)$
9	Sleator and Tarjan [1983]	$O(nm \log n)$
10	Tarjan [1984]	$O(n^3)$
11	Gabow [1985]	$O(nm \log U)$
12	Goldberg [1985]	$O(n^3)$
13	Goldberg and Tarjan [1986]	$O(nm \log (n^2/m))$
14	Bertsekas [1986]	$O(n^3)$
15	Cheriyan and Maheshwari [1987]	$O(n^2\sqrt{m})$
16	Ahuja and Orlin [1987]	$O(nm + n^2 \log U)$
17	Ahuja, Orlin and Tarjan [1988]	(a) $O\left(nm + \frac{n^2 \log U}{\log \log U}\right)$
		(b) $O\left(nm + n^2\sqrt{\log U}\right)$
		(c) $O\left(nm \log\left(\frac{n\sqrt{\log U}}{m} + 2\right)\right)$

Table 6.2. Running times of maximum flow algorithms.

Ford and Fulkerson [1956] observed that the labeling algorithm can perform as many as O(nU) augmentations for networks with integer arc capacities. They also showed that for arbitrary irrational arc capacities, the labeling algorithm can perform an infinite sequence of augmentations and might converge to a value different from the maximum flow value. Edmonds and Karp [1972] suggested two specializations of the labeling algorithm, both with improved computational complexity. They showed that if the algorithm augments flow along a shortest path (i.e., one containing the smallest possible number of arcs) in the residual network, then the algorithm performs O(nm) augmentations. A breadth first search of the network will determine a shortest augmenting path; consequently, this version of the labeling

algorithm runs in $O(nm^2)$ time. Edmonds and Karp's second idea was to augment flow along a path with maximum residual capacity. They proved that this algorithm performs O(m log U) augmentations. Tarjan [1986] has shown how to determine a path with maximum residual capacity in O(m) time on average; hence, this version of the labeling algorithm runs in $O(m^2 \log U)$ time.

Dinic [1970] independently introduced the concept of shortest path networks, called *layered networks* , for solving the maximum flow problem. A layered network is a subgraph of the residual network that contains only those nodes and arcs that lie on at least one shortest path from the source to the sink. The nodes in a layered network can be partitioned into layers of nodes $N_1, N_2, \ldots$, so that for every arc (i, j) in the layered network connects nodes in adjacent layers (i.e., , $i \in N_k$ and $j \in N_{k+1}$ for some k). A *blocking flow* in a layered network G' = (N', A') is a flow that blocks flow augmentations in the sense that G' contains no directed path with positive residual capacity from the source node to the sink node. Dinic showed how to construct, in a total of O(nm) time, a blocking flow in a layered network by performing at most m augmentations. His algorithm constructs layered networks and establishes blocking flows in these networks. Dinic showed that after each blocking flow iteration, the length of the layered network increases and after at most n iterations, the source is disconnected from the sink in the residual network. Consequently, his algorithm runs in $O(n^2m)$ times.

The shortest augmenting path algorithm presented in Section 4.3 achieves the same time bound as Dinic's algorithm, but instead of constructing layered networks it maintains distance labels. Goldberg [1985] introduced distance labels in the context of his preflow push algorithm. Distance labels offer several advantages: They are simpler to understand than layered networks, are easier to manipulate, and have led to more efficient algorithms. Orlin and Ahuja [1987] developed the distance label based augmenting path algorithm given in Section 4.3. They also showed that this algorithm is equivalent both to Edmonds and Karp's algorithm and to Dinic's algorithm in the sense that all three algorithms enumerate the same augmenting paths in the same sequence. The algorithms differ only in the manner in which they obtain these augmenting paths.

Several researchers have contributed improvements to the computational complexity of maximum flow algorithms by developing more efficient algorithms to establish blocking flows in layered networks. Karzanov [1974] introduced the concept

of preflows in a layered network. (See the technical report of Even [1976] for a comprehensive description of this algorithm and the paper by Tarjan [1984] for a simplified version.) Karzanov showed that an implementation that maintains preflows and pushes flows from nodes with excesses, constructs a blocking flow in $O(n^2)$ time. Malhotra, Kumar and Maheshwari [1978] present a conceptually simple maximum flow algorithm that runs in $O(n^3)$ time. Cherkasky [1977] and Galil [1980] presented further improvements of Karzanov's algorithm.

The search for more efficient maximum flow algorithms has stimulated researchers to develop new data structure for implementing Dinic's algorithm. The first such data structures were suggested independently by Shiloach [1978] and Galil and Naamad [1980]. Dinic's algorithm (or the shortest augmenting path algorithm described in Section 4.3) takes O(n) time on average to identify an augmenting path and, during the augmentation, it saturates some arcs in this path. If we delete the saturated arcs from this path, we obtain a set of *path fragments*. The basic idea is to store these path fragments using some data structure, for example, 2-3 trees (see Aho, Hopcroft and Ullman [1974] for a discussion of 2-3 trees) and use them later to identify augmenting paths quickly. Shiloach [1978] and Galil and Naamad [1980] showed how to augment flows through path fragments in a way that finds a blocking flow in $O(m(\log n)^2)$ time. Hence, their implementation of Dinic's algorithm runs in $O(nm (\log n)^2)$ time. Sleator and Tarjan [1983] improved this approach by using a data structure called *dynamic trees* to store and update path fragments. Sleator and Tarjan's algorithm establishes a blocking flow in O(m log n) time and thereby yields an O(nm log n) time bound for Dinic's algorithm.

Gabow [1985] obtained a similar time bound by applying a bit scaling approach to the maximum flow problem. As outlined in Section 1.7, this approach solves a maximum flow problem at each scaling phase with one more bit of every arc's capacity. During a scaling phase, the initial flow value differs from the maximum flow value by at most m units and so the shortest augmenting path algorithm (and also Dinic's algorithm) performs at most m augmentations. Consequently, each scaling phase takes O(nm) time and the algorithm runs in O(nm log C) time. If we invoke the similarity assumption, this time bound is comparable to that of Sleator and Tarjan's algorithm, but the scaling algorithm is much simpler to implement. Orlin and Ahuja [1987] have presented a variation of Gabow's algorithm achieving the same time bound.

Goldberg and Tarjan [1986] developed the generic preflow push algorithm and the highest-label preflow push algorithm. Previously, Goldberg [1985] had shown that the FIFO version of the algorithm that pushes flow from active nodes in the first-in-first-out order runs in $O(n^3)$ time. (This algorithm maintains a *queue* of active nodes; at each iteration, it selects a node from the front of the queue, performs a push/relabel step at this node, and adds the newly active nodes to the rear of the queue.) Using a dynamic tree data structure, Goldberg and Tarjan [1986] improved the running time of the FIFO preflow push algorithm to $O(nm \log (n^2/m))$. This algorithm currently gives the best strongly polynomial-time bound for solving the maximum flow problem.

Bertsekas [1986] obtained another maximum flow algorithm by specializing his minimum cost flow algorithm; this algorithm closely resembles the Goldberg's FIFO preflow push algorithm. Recently, Cheriyan and Maheshwari [1987] showed that Goldberg and Tarjan's highest-label preflow push algorithm actually performs $O(n^2\sqrt{m})$ nonsaturating pushes and hence runs in $O(n^2\sqrt{m})$ time.

Ahuja and Orlin [1987] improved the Goldberg and Tarjan's algorithm using the excess-scaling technique to obtain an $O(nm + n^2 \log U)$ time bound. If we invoke the similarity assumption, this algorithm improves Goldberg and Tarjan's $O(nm \log (n^2/m))$ algorithm by a factor of log n for networks that are both non-sparse and nondense. Further, this algorithm does not use any complex data structures. Scaling excesses by a factor of log U/log log U and pushing flow from a large excess node with the highest distance label, Ahuja, Orlin and Tarjan [1988] reduced the number of nonsaturating pushes to $O(n^2 \log U/ \log \log U)$. Ahuja, Orlin and Tarjan [1988] obtained another variation of original excess scaling algorithm which further reduces the number of nonsaturating pushes to $O(n^2 \sqrt{\log U})$.

The use of the dynamic tree data structure improves the running times of the excess-scaling algorithm and its variations, though the improvements are not as dramatic as they have been for Dinic's and the FIFO preflow push algorithms. For example, the $O(nm + n^2 \sqrt{\log U})$ algorithm improves to $O\left(nm \log\left(\frac{n\sqrt{\log U}}{m} + 2\right)\right)$ by using dynamic trees, as shown in Ahuja, Orlin and Tarjan [1988]. Tarjan [1987] conjectures that any preflow push algorithm that performs p nonsaturating pushes can be implemented in $O(nm \log (2+p/nm))$ time using dynamic trees. Although this

conjecture is true for all known preflow push algorithms, it is still open for the general case.

Developing a polynomial-time primal simplex algorithm for the maximum flow problem has been an outstanding open problem for quite some time. Recently, Goldfarb and Hao [1988] developed such an algorithm. This algorithm is essentially based on selecting pivot arcs so that flow is augmented along a shortest path from the source to the sink. As one would expect, this algorithm performs O(nm) pivots and can be implemented in $O(n^2m)$ time. Tarjan[1988] recently showed how to implement this algorithm in O(nm logn) using dynamic trees.

Researchers have also investigated the following special cases of the maximum flow problems: the maximum flow problem on (i) unit capacity networks (i.e., U=1); (ii) unit capacity simple networks (i.e., U=1, and, every node in the network, except source and sink, has one incoming arc or one outgoing arc) ; (iii) bipartite networks; and (iv) planar networks. Observe that the maximum flow value for unit capacity networks is less than n, and so the shortest augmenting path algorithm will solve these problems in O(nm) time. Thus, these problems are easier to solve than are problems with large capacities. Even and Tarjan [1975] showed that Dinic's algorithm solves the maximum flow problem on unit capacity networks in $O(n^{2/3}m)$ time and on unit capacity simple networks in $O(n^{1/2}m)$ time. Orlin and Ahuja [1987] have achieved the same time bounds using a modification of the shortest augmenting path algorithm. Both of these algorithms rely on ideas contained in Hopcraft and Karp's [1973] algorithm for maximum bipartite matching. Fernandez-Baca and Martel [1987] have generalized these ideas for networks with small integer capacities.

Versions of the maximum flow algorithms run considerably faster on a bipartite networks $G = (N_1 \cup N_2, A)$ if $|N_1| << |N_2|$ (or $|N_2| << |N_1|$). Let $n_1 = |N_1|$, $n_2 = |N_2|$ and $n = n_1 + n_2$. Suppose that $n_1 \leq n_2$. Gusfield, Martel and Fernandez-Baca [1985] obtained the first such results by showing how the running times of Karzanov's and Malhotra et al.'s algorithms reduce from $O(n^3)$ to $O(n_1^2 n_2)$ and $O(n_1^3 + nm)$ respectively. Ahuja, Orlin, Stein and Tarjan [1988] improved upon these ideas by showing that it is possible to substitute n_1 for n in the time bounds for all preflow push algorithms to obtain the new time bounds for bipartite networks. This result implies that the FIFO preflow push algorithm and the

original excess scaling algorithm, respectively, solve the maximum flow problem on bipartite networks in $O(n_1 m + n_1^3)$ and $O(n_1 m + n_1^2 \log U)$ time.

It is possible to solve the maximum flow problem on planar networks much more efficiently than on general networks. (A network is called *planar* if it can be drawn in a two-dimensional plane so that arcs intersect one another only at the nodes.) A planar network has at most 6n arcs; hence, the running times of the maximum flow algorithms on planar networks appear more attractive. Specialized solution techniques, that have even better running times, are quite different than those for the general networks. Some important references for planar maximum flow algorithms are Itai and Shiloach [1979], Johnson and Venkatesan [1982] and Hassin and Johnson [1985].

Researchers have also investigated whether the worst-case bounds of the maximum flow algorithms are *tight*, i.e., whether the algorithms achieve their worst-case bounds for some families of networks. Zadeh [1972] showed that the bound of Edmonds and Karp algorithm is tight when $m = n^2$. Even and Tarjan [1975] noted that the same examples imply that the bound of Dinic's algorithm is tight when $m = n^2$. Baratz [1977] showed that the bound on Karzanov's algorithm is tight. Galil [1981] constructed an interesting class of examples and showed that the algorithms of Edmonds and Karp, Dinic, Karzanov, Cherkasky, Galil and Malhotra et al. achieve their worst-case bounds on those examples.

Other researchers have made some progress in constructing worst-case examples for preflow push algorithms. Martel [1987] showed that the FIFO preflow push algorithm can take $\Omega(nm)$ time to solve a class of unit capacity networks. Cheriyan and Maheshwari [1987] have shown that the bound of $O(n^2 \sqrt{m})$ for the highest-label preflow push algorithm is tight. Cheriyan [1988] has also constructed a family of examples to show that the bound $O(n^3)$ for FIFO preflow push algorithm and the bound $O(n^2m)$ for the generic preflow push algorithm is tight. The research community has not established similar results for other preflow push algorithms, especially for the excess-scaling algorithms. It is worth mentioning, however, that these known worst-case examples are quite artificial and are not likely to arise in practice.

Several computational studies have assessed the empirical behavior of maximum flow algorithms. The studies performed by Hamacher [1979], Cheung

[1980], Glover, Klingman, Mote and Whitman [1979, 1984], Imai [1983] and Goldfarb and Grigoriadis [1986] are noteworthy. These studies were conducted prior to the development of algorithms that use distance labels. These studies rank Edmonds and Karp, Dinic's and Karzanov's algorithms in increasing order of performance for most classes of networks. Dinic's algorithm is competitive with Karzanov's algorithm for sparse networks, but slower for dense networks. Imai [1983] noted that Galil and Naamad's [1980] implementation of Dinic's algorithm, using sophisticated data structures, is slower than the original Dinic's algorithm. Sleator and Tarjan [1983] reported a similar finding; they observed that their implementation of Dinic's algorithm using dynamic tree data structure is slower than the original Dinic's algorithm by a constant factor. Hence, the sophisticated data structures improve only the worst-case performance of algorithms, but are not useful empirically. Researchers have also tested the Malhotra et al. algorithm and the primal simplex algorithm due to Fulkerson and Dantzig [1955] and found these algorithms to be slower than Dinic's algorithm for most classes of networks.

A number of researchers are currently evaluating the computational performance of preflow push algorithms. Derigs and Meier [1988], Grigoriadis [1988], and Ahuja, Kodialam and Orlin [1988] have found that the preflow push algorithms are substantially (often 2 to 10 times) faster than Dinic's and Karzanov's algorithms for most classes of networks. Among all nonscaling preflow push algorithms, the highest-label preflow push algorithm runs the fastest. The excess-scaling algorithm and its variations have not been tested thoroughly. We do not anticipate that dynamic tree implementations of preflow push algorithms would be useful in practice; in this case, as in others, their contribution has been to improve the worst-case performances of algorithms.

Finally, we discuss two important generalizations of the maximum flow problem: (i) the multi-terminal flow problem; (ii) the maximum dynamic flow problem.

In the multi-terminal flow problem, we wish to determine the maximum flow value between every pair of nodes. Gomory and Hu [1961] showed how to solve the multi-terminal flow problem on undirected networks by solving (n-1) maximum flow problems. Recently, Gusfield [1987] has suggested a simpler multi-terminal flow algorithm. These results, however , do not apply to the multi-terminal maximum flow problem on directed networks.

In the simplest version of maximum dynamic flow problem, we associate with each arc (i, j) in the network a number t_{ij} denoting the time needed to traverse that arc. The objective is to send the maximum possible flow from the source node to the sink node within a given time period T. Ford and Fulkerson [1958] first showed that the maximum dynamic flow problem can be solved by solving a minimum cost flow problem. (Ford and Fulkerson [1962] give a nice treatment of this problem). Orlin [1983] has considered infinite horizon dynamic flow problems in which the objective is to minimize the average cost per period.

6.4 Minimum Cost Flow Problem

The minimum cost flow problem has a rich history. The classical transportation problem, a special case of the minimum cost flow problem,was posed and solved (though incompletely) by Kantorovich [1939], Hitchcock [1941], and Koopmans [1947]. Dantzig [1951] developed the first complete solution procedure for the transportation problem by specializing his simplex algorithm for linear programming. He observed the spanning tree property of the basis and the integrality property of the optimum solution. Later his development of the upper bounding technique for linear programming led to an efficient specialization of the simplex algorithm for the minimum cost flow problem. Dantzig's book [1962] discusses these topics.

Ford and Fulkerson [1956, 1957] suggested the first combinatorial algorithms for the uncapacitated and capacitated transportation problem; these algorithms are known as the primal-dual algorithms. Ford and Fulkerson [1962] describe the primal-dual algorithm for the minimum cost flow problem. Jewell [1958], Iri [1960] and Busaker and Gowen [1961] independently discovered the successive shortest path algorithm. These researchers showed how to solve the minimum cost flow problem as a sequence of shortest path problems with arbitrary arc lengths. Tomizava [1971] and Edmonds and Karp [1972] independently pointed out that if the computations use node potentials, then these algorithms can be implemented so that the shortest path problems have nonnegative arc lengths.

Minty [1960] and Fulkerson [1961] independently discovered the out-of-kilter algorithm. The negative cycle algorithm is credited to Klein [1967]. Helgason and Kennington [1977] and Armstrong, Klingman and Whitman [1980] describe the

specialization of the linear programming dual simplex algorithm for the minimum cost flow problem (which is not discussed in this chapter). Each of these algorithms perform iterations that can (apparently) not be polynomially bounded. Zadeh [1973a] describes one such example on which each of several algorithms -- the primal simplex algorithm with Dantzig's pivot rule, the dual simplex algorithm, the negative cycle algorithm (which augments flow along a most negative cycle), the successive shortest path algorithm, the primal-dual algorithm, and the out-of-kilter algorithm -- performs an exponential number of iterations. Zadeh [1973b] has also described more pathological examples for network algorithms.

The fact that one example is bad for many network algorithms suggests inter-relationship among the algorithms. The insightful paper by Zadeh [1979] showed this relationship by pointing out that each of the algorithms just mentioned are indeed equivalent in the sense that they perform the same sequence of augmentations provided ties are broken using the same rule. All these algorithms essentially consist of identifying shortest paths between appropriately defined nodes and augmenting flow along these paths. Further, these algorithms obtain shortest paths using a method that can be regarded as an application of Dijkstra's algorithm.

The network simplex algorithm and its practical implementations have been most popular with operations researchers. Johnson [1966] suggested the first tree manipulating data structure for implementing the simplex algorithm. The first implementations using these ideas, due to Srinivasan and Thompson [1973] and Glover, Karney, Klingman and Napier [1974], significantly reduced the running time of the simplex algorithm. Glover, Klingman and Stutz [1974], Bradley, Brown and Graves [1977], and Barr, Glover, and Klingman [1979] subsequently discovered improved data structures. The book of Kennington and Helgason [1980] is an excellent source for references and background material concerning these developements.

Researchers have conducted extensive studies to determine the most effective pricing strategy, i.e., selection of the entering variable. These studies show that the choice of the pricing strategy has a significant effect on both solution time and the number of pivots required to solve minimum cost flow problems. The candidate list strategy we described is due to Mulvey [1978a]. Goldfarb and Reid [1977], Bradley, Brown and Graves [1978], Grigoriadis and Hsu [1979], Gibby, Glover, Klingman and Mead [1983] and Grigoriadis [1986] have described other strategies that have been

effective in practice. It appears that the best pricing strategy depends both upon the network structure and the network size.

Experience with solving large scale minimum cost flow problems has established that more than 90% of the pivoting steps in the simplex method can be degenerate (see Bradley, Brown and Graves [1978], Gavish, Schweitzer and Shlifer [1977] and Grigoriadis [1986]). Thus, degeneracy is both a computational and a theoretical issue. The strongly feasible basis technique, proposed by Cunningham [1976] and independently by Barr, Glover and Klingman [1977a, 1977b, 1978] has contributed on both fronts. Computational experience has shown that maintaining strongly feasible basis substantially reduces the number of degenerate pivots. On the theoretical front, the use of this technique led to a finitely converging primal simplex algorithm. Orlin [1985] showed, using a perturbation technique, that for integer data an implementation of the primal simplex algorithm that maintains a strongly feasible basis performs O(nmCU) pivots when used with any arbitrary pricing strategy and O(nm C log (mCU)) pivots when used with Dantzig's pricing strategy.

The strongly feasible basis technique prevents cycling during a sequence of consecutive degenerate pivots, but the number of consecutive degenerate pivots may be exponential. This phenomenon is known as *stalling*. Cunningham [1979] described an example of stalling and suggested several rules for selecting the entering variable to avoid stalling. One such rule is the LRC (Least Recently Considered) rule which orders the arcs in an arbitrary, but fixed, manner. The algorithm then examines the arcs in the wrap-around fashion, each iteration starting at a place where it left off earlier, and introduces the first eligible arc into the basis. Cunningham showed that this rule admits at most nm consecutive degenerate pivots. Goldfarb, Hao and Kai [1987] have described more anti-stalling pivot rules for the minimum cost flow problem.

Researchers have also been interested in developing polynomial-time simplex algorithms for the minimum cost flow problem or its special cases. The only polynomial time-simplex algorithm for the minimum cost flow problem is a dual simplex algorithm due to Orlin [1984]; this algorithm performs $O(n^3 \log n)$ pivots for the uncapacitated minimum cost flow problem. Developing a polynomial-time primal simplex algorithm for the minimum cost flow problem is still open. However, researchers have developed such algorithms for the shortest path problem, the maximum flow problem, and the assignment problem: Dial et al. [1979], Zadeh

[1979], Orlin [1985], Akgul [1985a], Goldfarb, Hao and Kai [1986] and Ahuja and Orlin [1988] for the shortest path problem; Goldfarb and Hao [1988] for the maximum flow problem; and Roohy-Laleh [1980], Hung [1983], Orlin [1985], Akgul [1985b] and Ahuja and Orlin [1988] for the assignment problem.

The *relaxation algorithms* proposed by Bertsekas and his associates are other attractive algorithms for solving the minimum cost flow problem and its generalization. For the minimum cost flow problem, this algorithm maintains a pseudoflow satisfying the optimality conditions. The algorithm proceeds by either (i) augmenting flow from an excess node to a deficit node along a path consisting of arcs with zero reduced cost, or (ii) changing the potentials of a subset of nodes. In the latter case, it resets flows on some arcs to their lower or upper bounds so as to satisfy the optimality conditions; however, this flow assignment might change the excesses and deficits at nodes. The algorithm operates so that each change in the node potentials increases the dual objective function value and when it finally determines the optimum dual objective function value, it has also obtained an optimum primal solution. This relaxation algorithm has exhibited nice empirical behavior. Bertsekas [1985] suggested the relaxation algorithm for the minimum cost flow problem (with integer data). Bertsekas and Tseng [1985] extended this approach for the minimum cost flow problem with real data, and for the generalized minimum cost flow problem (see Section 6.6 for a definition of this problem).

A number of empirical studies have extensively tested minimum cost flow algorithms for wide variety of network structures, data distributions, and problem sizes. The most common problem generator is NETGEN, due to Klingman, Napier and Stutz [1974], which is capable of generating assignment, and capacitated or uncapacitated transportation and minimum cost flow problems. Glover, Karney and Klingman [1974] and Aashtiani and Magnanti [1976] have tested the primal-dual and out-of-kilter algorithms. Helgason and Kennington [1977] and Armstrong, Klingman and Whitman [1980] have reported on extensive studies of the dual simplex algorithm. The primal simplex algorithm has been a subject of more rigorous investigation; studies conducted by Glover, Karney, Klingman and Napier [1974] , Glover, Karney and Klingman [1974], Bradley, Brown and Graves [1977], Mulvey [1978b], Grigoriadis and Hsu [1979] and Grigoriadis [1986] are noteworthy. Bertsekas and Tseng [1988] have presented computational results for the relaxation algorithm.

In view of Zadeh's [1979] result, we would expect that the successive shortest path algorithm, the primal-dual algorithm, the out-of-kilter algorithm, the dual simplex algorithm, and the primal simplex algorithm with Dantzig's pivot rule should have comparable running times. By using more effective pricing strategies that determine a good entering arc without examining all arcs, we would expect that the primal simplex algorithm should outperform other algorithms. All the computational studies have verified this expectation and until very recently the primal simplex algorithm has been a clear winner for almost all classes of network problems. Bertsekas and Tseng [1988] have reported that their relaxation algorithm is substantially faster than the primal simplex algorithm. However, Grigoriadis [1986] finds his new version of primal simplex algorithm faster than the relaxation algorithm. At this time, it appears that the relaxation algorithm of Bertsekas and Tseng, and the primal simplex algorithm due to Grigoriadis are the two fastest algorithms for solving the minimum cost flow problem in practice.

Computer codes for some minimum cost flow problem are available in the public domain. These include the primal simplex codes RNET and NETFLOW developed by Grigoradis and Hsu [1979] and Kennington and Helgason [1980], respectively, and the relaxation code RELAX developed by Bertsekas and Tseng [1988].

Polynomial-Time Algorithms

In the recent past, researchers have actively pursued the design of fast (weakly) polynomial and strongly polynomial-time algorithms for the minimum cost flow problem. Recall that an algorithm is strongly polynomial-time if its running time is polynomial in the number of nodes and arcs, and does not evolve terms containing logarithms of C or U. The table given in Figure 6.3 summarizes these theoretical developments in solving the minimum cost flow problem. The table reports running times for networks with n nodes and m arcs, m' of which are capacitated. It assumes that the integral cost coefficients are bounded in absolute value by C, and the integral capacities, supplies and demands are bounded in absolute value by U. The term S(·) is the running time for the shortest path problem and the term M(·) represents the corresponding running time to solve a maximum flow problem.

Polynomial-Time Combinatorial Algorithms

#	Discoverers	Running Time
1	Edmonds and Karp [1972]	$O((n + m') \log U\ S(n, m, C))$
2	Rock [1980]	$O((n + m') \log U\ S(n, m, C))$
3	Rock [1980]	$O(n \log C\ M(n, m, U))$
4	Bland and Jensen [1985]	$O(n \log C\ M(n, m, U))$
5	Goldberg and Tarjan [1988a]	$O(nm \log (n^2/m) \log nC)$
6	Bertsekas and Eckstein [1988]	$O(n^3 \log nC)$
7	Goldberg and Tarjan [1987]	$O(n^3 \log nC)$
7	Gabow and Tarjan [1987]	$O(nm \log n \log U \log nC)$
8	Goldberg and Tarjan [1987, 1988b]	$O(nm \log n \log nC)$
9	Ahuja, Goldberg, Orlin and Tarjan [1988]	$O(nm (\log U/\log \log U) \log nC)$ and $O(nm \log \log U \log nC)$

Strongly Polynomial -Time Combinatorial Algorithms

#	Discoverers	Running Time
1	Tardos [1985]	$O(m^4)$
2	Orlin [1984]	$O((n + m')^2\ S(n, m))$
3	Fujishige [1986]	$O((n + m')^2\ S(n, m))$
4	Galil and Tardos [1986]	$O(n^2 \log n\ S(n,m))$
5	Goldberg and Tarjan [1988a]	$O(nm^2 \log n\ \log(n^2/m))$
6	Goldberg and Tarjan [1988b]	$O(nm^2 \log^2 n)$
7	Orlin [1988]	$O((n + m') \log n\ S(n, m))$

Table 6.3 Polynomial-time algorithms for the minimum cost flow problem.

For the sake of comparing the polynomial and strongly polynomial-time algorithms, we invoke the similarity assumption. For problems that satisfy the similarity assumption, the best bounds for the shortest path and maximum flow problems are:

Polynomial-Time Bounds	**Discoverers**
$S(n,m, C) = \min \{m \log\log C, m + n\sqrt{\log C}\}$	Johnson [1982], and Ahuja, Mehlhorn, Orlin and Tarjan [1988]
$M(n, m, C) = nm \log\left(\frac{n\sqrt{\log U}}{m} + 2\right)$	Ahuja, Orlin and Tarjan [1987]

Strongly Polynomial -Time Bounds	**Discoverers**
$S(n, m) = m + n \log n$	Fredman and Tarjan [1984]
$M(n, m) = nm \log (n^2/m)$	Goldberg and Tarjan [1986]

Using capacity and right-hand-side scaling, Edmonds and Karp [1972] developed the first (weakly) polynomial-time algorithm for the minimum cost flow problem. The RHS-scaling algorithm presented in Section 5.7, which is a variant of the Edmonds-Karp algorithm, was suggested by Orlin [1988]. The scaling technique did not initially capture the interest of many researchers, since they regarded it as having little practical utility. However, researchers gradually recognized that the scaling technique has great theoretical value as well as potential practical significance. Rock [1980] developed two different bit-scaling algorithms for the minimum cost flow problem, one using capacity scaling and the other using cost scaling. This cost scaling algorithm reduces the minimum cost flow problem to a sequence of O(n log C) maximum flow problems. Bland and Jensen [1985] independently discovered a similar cost scaling algorithm.

The pseudoflow push algorithms for the minimum cost flow problem discussed in Section 5.8 use the concept of *approximate optimality,* introduced independently by Bertsekas [1979] and Tardos [1985]. Bertsekas [1986] developed the first pseudoflow push algorithm. This algorithm was pseudopolynomial-time. Goldberg and Tarjan [1987] used a scaling technique on a variant of this algorithm to obtain the generic pseudoflow push algorithm described in Section 5.8. Tarjan [1984] proposed a wave algorithm for the maximum flow problem. The wave algorithm

for the minimum cost flow problem described in Section 5.8 , which was developed independently by Goldberg and Tarjan [1987] and Bertsekas and Eckstein [1988], relies upon similar ideas. Using a dynamic tree data structure in the generic pseudoflow push algorithm, Goldberg and Tarjan [1987] obtained a computational time bound of O(nm log n log nC). They also showed that the minimum cost flow problem can be solved using O(n log nC) blocking flow computations. (The description of Dinic's algorithm in Section 6.3 contains the definition of a blocking flow.) Using both *finger tree* (see Mehlhorn [1984]) and *dynamic tree* data structures, Goldberg and Tarjan [1988a] obtained an O(nm log (n^2/m) log nC) bound for the wave algorithm.

These algorithms, except the wave algorithm, required sophisticated data structures that impose a very high computational overhead. Although the wave algorithm is very practical, its worst-case running time is not very attractive. This situation has prompted researchers to investigate the possibility of improving the computational complexity of minimum cost flow algorithms without using any complex data structures. The first success in this direction was due to Gabow and Tarjan [1987], who developed a triple scaling algorithm running in time O(nm log n log U log nC). The second success was due to Ahuja, Goldberg, Orlin and Tarjan [1988], who developed the double scaling algorithm. The double scaling algorithm, as described in Section 5.9, runs in O(nm log U log nC) time. Scaling costs by an appropriately larger factor improves the algorithm to O(nm(log U/log log U) log nC) , and a dynamic tree implementation improves the bound further to O(nm log log U log nC). For problems satisfying the similarity assumption, the double scaling algorithm is faster than all other algorithms for all network topologies except for very dense networks; in these instances, algorithms by Goldberg and Tarjan appear more attractive.

Goldberg and Tarjan [1988b] and Barahona and Tardos [1987] have developed other polynomial-time algorithms. Both the algorithms are based on the negative cycle algorithm due to Klein [1967]. Goldberg and Tarjan [1988b] showed that if the negative cycle algorithm always augments along flow a minimum mean cycle (a cycle W for which $\sum_{(i,j) \in W} c_{ij} / |W|$ is minimum), then it is strongly polynomial-time. Goldberg and Tarjan described an implementation of this approach running in time O(nm(log n) min{log nC, m log n}). Barahona and Tardos [1987], analyzing an algorithm suggested by Weintraub [1974], showed that if the negative cycle algorithm

augments flow along a cycle with maximum improvement in the objective function, then it performs O(m log mCU) iterations. Since identifying a cycle with maximum improvement is difficult (i.e., NP-hard), they describe a method (based upon solving an auxiliary assignment problem) to determine a disjoint set of augmenting cycles with the property that augmenting flows along these cycles improves the flow cost by at least as much as augmenting flow along any single cycle. Their algorithm runs in $O(m^2 \log (mCU) S(n, m, C))$ time.

Edmonds and Karp [1972] proposed the first polynomial-time algorithm for the minimum cost flow problem, and also highlighted the desire to develop a strongly polynomial-time algorithm. This desire was motivated primarily by theoretical considerations. (Indeed, in practice, the terms log C and log U typically range from 1 to 20, and are sublinear in n.) Strongly polynomial-time algorithms are theoretically attractive for at least two reasons: (i) they might provide, in principle, network flow algorithms that can run on real valued data as well as integer valued data, and (ii) they might, at a more fundamental level, identify the source of the underlying complexity in solving a problem; i.e., are problems more difficult or equally difficult to solve as the values of the underlying data becomes increasingly larger?

The first strongly polynomial-time minimum cost flow algorithm is due to Tardos [1985]. Several researchers including Orlin [1984], Fujishige [1986], Galil and Tardos [1986], and Orlin [1988] provided subsequent improvements in the running time. Goldberg and Tarjan [1988a] obtained another strongly polynomial time algorithm by slightly modifying their pseudoflow push algorithm. Goldberg and Tarjan [1988b] also show that their algorithm that proceeds by cancelling minimum mean cycles is also strongly polynomial time. Currently, the fastest strongly polynomial-time algorithm is due to Orlin [1988]. This algorithm solves the minimum cost flow problem as a sequence of O(min(m log U, m log n)) shortest path problems. For very sparse networks, the worst-case running time of this algorithm is nearly as low as the best weakly polynomial-time algorithm, even for problems that satisfy the similarity assumption.

Interior point linear programming algorithms are another source of polynomial-time algorithms for the minimum cost flow problem. Kapoor and Vaidya [1986] have shown that Karmarkar's [1984] algorithm, when applied to the minimum cost flow problem performs $O(n^{2.5} mK)$ operations, where

$K = \log n + \log C + \log U$. Vaidya [1986] suggested another algorithm for linear programming that solves the minimum cost flow problem in $O(n^{2.5} \sqrt{m}\ K)$ time. Asymptotically, these time bounds are worse than that of the double scaling algorithm.

At this time, the research community has yet to develop sufficient evidence to fully assess the computational worth of scaling and interior point linear programming algorithms for the minimum cost flow problem. According to the folklore, even though they might provide the best-worst case bounds on running times, the scaling algorithms are not as efficient as the non-scaling algorithms. Boyd and Orlin [1986] have obtained contradictory results. Testing the right-hand-side scaling algorithm for the minimum cost flow problem, they found the scaling algorithm to be competitive with the relaxation algorithm for some classes of problems. Bland and Jensen [1985] also reported encouraging results with their cost scaling algorithm. We believe that when implemented with appropriate speed-up techniques, scaling algorithms have the potential to be competitive with the best other algorithms.

6.5 Assignment Problem

The assignment problem has been a popular research topic. The primary emphasis in the literature has been on the development of empirically efficient algorithms rather than the development of algorithms with improved worst-case complexity. Although the research community has developed several different algorithms for the assignment problem, many of these algorithms share common features. The successive shortest path algorithm, described in Section 5.4 for the minimum cost flow problem, appears to lie at the heart of many assignment algorithms. This algorithm is implicit in the first assignment algorithm due to Kuhn [1955], known as the *Hungarian method*, and is explicit in the papers by Tomizava [1971] and Edmonds and Karp [1972].

When applied to an assignment problem on the network $G = (N_1 \cup N_2, A)$, the successive shortest path algorithm operates as follows. To use this solution approach, we first transform the assignment problem into a minimum cost flow problem by adding a source node s and a sink node t, and introducing arcs (s,i) for all $i \in N_1$, and (j,t) for all $j \in N_2$; these arcs have zero cost and unit capacity. The algorithm successively obtains a shortest path from s to t with respect to the linear

programming reduced costs, updates the node potentials, and augments one unit of flow along the shortest path. The algorithm solves the assignment problem by n applications of the shortest path algorithm for nonnegative arc lengths and runs in O(nS(n,m,C)) time, where S(n,m,C) is the time needed to solve a shortest path problem. For a naive implementation of Dijkstra's algorithm, S(n,m,C) is $O(n^2)$ and for a Fibonacci heap implementation it is O(m+nlogn). For problems satisfying the similarity assumption, S(n,m,C) is min{m log log C, $m+n\sqrt{\log C}$}.

The fact that the assignment problem can be solved as a sequence of n shortest path problems with arbitrary arc lengths follows from the works of Jewell [1958], Iri [1960] and Busaker and Gowen [1961] on the minimum cost flow problem. However, Tomizava [1971] and Edmonds and Karp [1972] independently pointed out that working with reduced costs leads to shortest path problems with nonnegative arc lengths. Weintraub and Barahona [1979] worked out the details of Edmonds-Karp algorithm for the assignment problem. The more recent *threshold assignment algorithm* by Glover, Glover and Klingman [1986] is also a successive shortest path algorithm which integrates their threshold shortest path algorithm (see Glover, Glover and Klingman [1984]) with the flow augmentation process. Carraresi and Sodini [1986] also suggested a similar threshold assignment algorithm.

Hoffman and Markowitz [1963] pointed out the transformation of a shortest path problem to an assignment problem.

Kuhn's [1955] Hungarian method is the primal-dual version of the successive shortest path algorithm. After solving a shortest path problem and updating the node potentials, the Hungarian method solves a (particularly simple) maximum flow problem to send the maximum possible flow from the source node s to the sink node t using arcs with zero reduced cost. Whereas the successive shortest path problem augments flow along one path in an iteration, the Hungarian method augments flow along all the shortest paths from the source node to the sink node. If we use the labeling algorithm to solve the resulting maximum flow problems, then these applications take a total of O(nm) time overall, since there are n augmentations and each augmentation takes O(m) time. Consequently, the Hungarian method, too, runs in O(nm + nS(n,mC)) = O(nS(n,m,C)) time. (For some time after the development of the Hungarian method as described by Kuhn, the research community considered it to be $O(n^4)$ method. Lawler [1976] described an $O(n^3)$

implementation of the method. Subsequently, many researchers realized that the Hungarian method in fact runs in O(nS(n,m,C)) time.) Jonker and Volgenant [1986] suggested some practical improvements of the Hungarian method.

The relaxation approach for the minimum cost flow problem is due to Dinic and Kronrod [1969], Hung and Rom [1980] and Engquist [1982]. This approach is closely related to the successive shortest path algorithm. Both approaches start with an infeasible assignment and gradually make it feasible. The major difference is in the nature of the infeasibility. The successive shortest path algorithm maintains a solution with unassigned persons and objects, and with no person or object overassigned. Throughout the relaxation algorithm, every person is assigned, but objects may be overassigned or unassigned. Both the algorithms maintain optimality of the intermediate solution and work toward feasibility by solving at most n shortest path problems with nonnegative arc lengths. The algorithms of Dinic and Kronrod [1969] and Engquist [1982] are essentially the same as the one we just described, but the shortest path computations are somewhat disguised in the paper of Dinic and Kronrod [1969]. The algorithm of Hung and Rom [1980] maintains a strongly feasible basis rooted at an overassigned node and, after each augmentation, reoptimizes over the previous basis to obtain another strongly feasible basis. All of these algorithms run in O(nS(n,m,C)) time.

Another algorithm worth mentioning is due to Balinski and Gomory [1964]. This algorithm is a primal algorithm that maintains a feasible assignment and gradually converts it into an optimum assignment by augmenting flows along negative cycles or by modifying node potentials. Derigs [1985] notes that the shortest path computations underlie this method, and that it runs in O(nS(n,m,C)) time.

Researchers have also studied primal simplex algorithms for the assignment problem. The basis of the assignment problem is highly degenerate; of its 2n-1 variables, only n are nonzero. Probably because of this excessive degeneracy, the mathematical programming community did not conduct much research on the network simplex method for the assignment problem until Barr, Glover and Klingman [1977a] devised the strongly feasible basis technique. These authors developed the details of the network simplex algorithm when implemented to maintain a strongly feasible basis for the assignment problem; they also reported encouraging computational results. Subsequent research focused on developing

polynomial-time simplex algorithms. Roohy-Laleh [1980] developed a simplex pivot rule requiring $O(n^3)$ pivots. Hung [1983] describes a pivot rule that performs at most $O(n^2)$ consecutive degenerate pivots and at most O(n log nC) nondegenerate pivots. Hence, his algorithm performs $O(n^3 \log nC)$ pivots. Akgul [1985b] suggested another primal simplex algorithm performing $O(n^2)$ pivots. This algorithm essentially amounts to solving n shortest path problems and runs in O(nS(n,m,C)) time.

Orlin [1985] studied the theoretical properties of Dantzig's pivot rule for the network simplex algorithm and showed that for the assignment problem this rule requires $O(n^2 \log nC)$ pivots. A naive implementation of the algorithm runs in $O(n^2 m \log nC)$. Ahuja and Orlin [1988] described a scaling version of Dantzig's pivot rule that performs $O(n^2 \log C)$ pivots and can be implemented to run in O(nm log C) time using simple data structures. The algorithm essentially consists of pivoting in any arc with *sufficiently large* reduced cost. The algorithm defines the term "sufficiently large" iteratively; initially, this threshold value equals C and within $O(n^2)$ pivots its value is halved.

Balinski [1985] developed the *signature method*, which is a dual simplex algorithm for the assignment problem. (Although his basic algorithm maintains a dual feasible basis, it is not a dual simplex algorithm in the traditional sense because it does not necessarily increase the dual objective at every iteration; some variants of this algorithm do have this property.) Balinski's algorithm performs $O(n^2)$ pivots and runs in $O(n^3)$ time. Goldfarb [1985] described some implementations of Balinski's algorithm that run in $O(n^3)$ time using simple data structures and in $O(nm + n^2 \log n)$ time using Fibonacci heaps.

The auction algorithm is due to Bertsekas and uses basic ideas originally suggested in Bertsekas [1979]. Bertsekas and Eckstein [1988] described a more recent version of the auction algorithm. Our presentation of the auction algorithm and its analysis is somewhat different that the one given by Bertsekas and Eckstein [1988]. For example, the algorithm we have presented increases the prices of the objects by one unit at a time, whereas the algorithm by Bertsekas and Eckstein increases prices by the maximum amount that preserves ε-optimality of the solution. Bertsekas [1981] has presented another algorithm for the assignment problem which is in fact a specialization of his relaxation algorithm for the minimum cost flow problem (see Bertsekas [1985]).

Currently, the best strongly polynomial-time bound to solve the assignment problem is $O(nm + n^2 \log n)$ which is achieved by many assignment algorithms. Scaling algorithms can do better for problems that satisfy the similarity assumption. Gabow [1985] , using bit-scaling of costs, developed the first scaling algorithm for the assignment problem. His algorithm performs $O(\log C)$ scaling phases and solves each phase in $O(n^{3/4}m)$ time, thereby achieving an $O(n^{3/4}m \log C)$ time bound. Using the concept of ϵ-optimality, Gabow and Tarjan [1987] developed another scaling algorithm running in time $O(n^{1/2}m \log nC)$. Observe that the generic pseudoflow push algorithm for the minimum cost flow problem described in Section 5.8 solves the assignment problem in $O(nm \log nC)$ since every push is a saturating push. Bertsekas and Eckstein [1988] showed that the scaling version of the auction algorithm runs in $O(nm \log nC)$. Section 5.11 has presented a modified version of this algorithm in Orlin and Ahuja [1988]. They also improved the time bound of the auction algorithm to $O(n^{1/2}m \log nC)$. This time bound is comparable to that of Gabow and Tarjan's algorithm, but the two algorithms would probably have different computational attributes. For problems satisfying the similarity assumption, these two algorithms achieve the best time bound to solve the assignment problem without using any sophisticated data structure.

As mentioned previously, most of the research effort devoted to assignment algorithms has stressed the development of empirically faster algorithms. Over the years, many computational studies have compared one algorithm with a few other algorithms. Some representative computational studies are those conducted by Barr, Glover and Klingman [1977a] on the network simplex method, by McGinnis [1983] and Carpento, Martello and Toth [1988] on the primal-dual method, by Engquist [1982] on the relaxation methods, and by Glover et al. [1986] and Jonker and Volgenant [1987] on the successive shortest path methods. Since no paper has compared all of these algorithms, it is difficult to assess their computational merits. Nevertheless, results to date seem to justify the following observations about the algorithms' relative performance. The primal simplex algorithm is slower than the primal-dual, relaxation and successive shortest path algorithms. Among the latter three approaches, the successive shortest path algorithms due to Glover et al. [1986] and Jonker and Volgenant [1987] appear to be the fastest. Bertsekas and Eckstein [1988] found that the scaling version of the auction algorithm is competitive with Jonker and Volgenant's algorithm. Carpento, Martello and Toth [1988] present

several FORTRAN implementations of assignment algorithms for dense and sparse cases.

6.6 Other Topics

Our discussion in this paper has featured single commodity network flow problems with linear costs. Several other generic topics in the broader problem domain of network optimization are of considerable theoretical and practical interest. In particular, four other topics deserve mention: (i) generalized network flows; (ii) convex cost flows; (iii) multicommodity flows; and (iv) network design. We shall now discuss these topics briefly.

Generalized Network Flows

The flow problems we have considered in this chapter assume that arcs conserve flows, i.e., the flow entering an arc equals the flow leaving the arc. In models of generalized network flows, arcs do not necessarily conserve flow. If x_{ij} units of flow enter an arc (i, j), then $r_{ij}\, x_{ij}$ units "arrive" at node j; r_{ij} is a nonnegative flow multiplier associated with the arc. If $0 < r_{ij} < 1$, then the arc is *lossy* and, if $1 < r_{ij} < \infty$, then the arc is *gainy*. In the conventional flow networks, $r_{ij} = 1$ for all arcs. Generalized network flows arise in may application contexts. For example, the multiplier might model pressure losses in a water resource network or losses incurred in the transportation of perishable goods.

Researchers have studied several generalized network flow problems. An extension of the conventional maximum flow problem is the *generalized maximum flow problem* which either maximizes the flow out of a source node or maximizes the flow into a sink node (these two objectives are different!) The source version of the problem can be states as the following linear program.

$$\text{Maximize } v_s \tag{6.1a}$$

subject to

$$\sum_{\{j:\,(i,j)\,\in\,A\}} x_{ij} \;-\; \sum_{\{j:\,(j,i)\,\in\,A\}} r_{ji}\, x_{ji} \;=\; \begin{cases} v_s, \text{ if } i = s \\ 0, \text{ if } i \neq s,t, \text{ for all } i \in N \\ v_t, \text{ if } i = t \end{cases} \tag{6.1b}$$

$0 \le x_{ij} \le u_{ij}$, for all $(i, j) \in A$.

Note that the capacity restrictions apply to the flows entering the arcs. Further, note that v_s is not necessarily equal to v_t, because of flow losses and gains within arcs.

The generalized maximum flow problem has many similarities with the minimum cost flow problem. Extended versions of the successive shortest path algorithm, the negative cycle algorithm, and the primal-dual algorithm for the minimum cost flow problem apply to the generalized maximum flow problem. The paper by Truemper [1977] surveys these approaches. These algorithms, however, are not pseudopolynomial-time, mainly because the optimal arc flows and node potentials might be fractional. The recent paper by Goldberg, Plotkin and Tardos [1986] describes the first polynomial-time combinatorial algorithms for the generalized maximum flow problem.

In the generalized minimum cost flow problem, which is an extension of the ordinary minimum cost flow problem, we wish to determine the minimum cost flow in a generalized network satisfying the specified supply/demand requirements of nodes. These are three main approaches to solve this problem. The first approach, due to Jewell [1982], is essentially a primal-dual algorithm. The second approach is the primal simplex algorithm studied by Elam, Glover and Klingman [1979] among others. Elam et al. find their implementation to be very efficient in practice; they find that it is about 2 to 3 times slower than their implementations for the ordinary minimum cost flow algorithm. The third approach, due to Bertsekas and Tseng [1988b], generalizes their minimum cost flow relaxation algorithm for the generalized minimum cost flow problem.

Convex Cost Flows

We shall restrict this brief discussion to convex cost flow problems with *separable* cost functions, i.e., the objective function can be written in the form $\sum_{(i,j) \in A} C_{ij}(x_{ij})$. Problems containing nonconvex nonseparable cost terms such as x_{12} x_{13} are substantially more difficult to solve and continue to pose a significant challenge for the mathematical programming community. Even problems with nonseparable, but convex objective functions are more difficult to solve; typically,

analysts rely on the general nonlinear programming techniques to solve these problems. The separable convex cost flow problem has the following formulation:

$$\text{Minimize} \quad \sum_{(i,j) \in A} C_{ij}(x_{ij}) \tag{6.2a}$$

subject to

$$\sum_{\{j:\,(i,j) \in A} x_{ij} - \sum_{\{j:\,(j,i) \in A} x_{ji} = b(i), \text{ for all } i \in N, \tag{6.2b}$$

$$0 \le x_{ij} \le u_{ij} \text{ , for all } (i, j) \in A. \tag{6.2c}$$

In this formulation, $C_{ij}(x_{ij})$ for each $(i,j) \in A$, is a convex function. The research community has focused on two classes of separable convex costs flow problems: (i) each $C_{ij}(x_{ij})$ is a piecewise linear function; of (ii) each $C_{ij}(x_{ij})$ is a continuously differentiable function. Solution techniques used to solve the two classes of problems are quite different.

There is a well-known technique for transforming a separable convex program with piecewise linear functions to a linear program (see, e.g., Bradley, Hax and Magnanti [1972]). This transformation reduces the convex cost flow problem to a standard minimum cost flow problem: it introduces one arc for each linear segment in the cost functions, thus increasing the problem size. However, it is possible to carry out this transformation implicitly and therefore modify many minimum cost flow algorithms such as the successive shortest path algorithm, negative cycle algorithm, primal-dual and out-of-kilter algorithms, to solve convex cost flow problems without increasing the problem size. The paper by Ahuja, Batra, and Gupta [1984] illustrates this technique and suggests a pseudopolynomial time algorithm.

Observe that it is possible to use a piecewise linear function, with linear segments chosen (if necessary) with sufficiently small size, to approximate a convex function of one variable to any desired degree of accuracy. More elaborate alternatives are possible. For example, if we knew the optimal solution to a separable convex problem a priori (which of course, we don't), then we could solve the problem *exactly* using a linear approximation for any arc (i, j) with only three

breakpoints: at 0, u_{ij} and the optimal flow on the arc. Any other breakpoint in the linear approximation would be irrelevant and adding other points would be computationally wasteful. This observation has prompted researchers to devise adaptive approximations that iteratively revise the linear approximation based upon the solution to a previous, coarser, approximation. (See Meyer [1979] for an example of this approach). If we were interested in only integer solutions, then we could choose the breakpoints of the linear approximation at the set of integer values, and therefore solve the problem in pseudopolynomial time.

Researchers have suggested other solution strategies, using ideas from nonlinear progamming for solving this general separable convex cost flow problems. Some important references on this topic are Ali, Helgason and Kennington [1978], Kennington and Helgason [1980], Meyer and Kao [1981], Dembo and Klincewicz [1981], Klincewicz [1983], Rockafellar [1984], Florian [1986], and Bertsekas, Hosein and Tseng [1987].

Some versions of the convex cost flow problems can be solved in polynomial time. Minoux [1984] has devised a polynomial-time algorithm for one of its special cases, the mininimum quadratic cost flow problem. Minoux [1986] has also developed a polynomial-time algorithm to obtain an *integer* optimum solution of the convex const flow problem.

Muticommodity Flows

Multicommodity flow problems arise when several commodities use the same underlying network, but share common arc capacities. In this section, we state a linear programming formulation of the multicommodity minimum cost flow problem and point the reader to contributions to this problem and its specializations.

Suppose that the problem contains r distinct commodities numbered 1 through r. Let b^k denote the supply/demand vector of commodity k. Then the multicommodity minimum cost flow problem can be formulated as follows:

$$\text{Minimize} \sum_{k=1}^{r} \sum_{(i,j)\in A} c_{ij}^{k} x_{ij}^{k} \qquad (6.3a)$$

subject to

$$\sum_{\{j:\,(i,j)\in A\}} x_{ij}^{k} - \sum_{\{j:\,(i,j)\in A\}} x_{ji}^{k} = b_i^{k}, \text{ for all } i \text{ and } k, \qquad (6.3b)$$

$$\sum_{k=1}^{r} x_{ij}^{k} \le u_{ij}, \text{ for all } (i,j), \qquad (6.3c)$$

$$0 \le x_{ij}^{k} \le u_{ij}^{k}, \text{ for all (i,j) and all k}. \qquad (6.3d)$$

In this formulation, x_{ij}^{k} and c_{ij}^{k} represent the amont of flow and the unit cost of flow for commodity k on arc (i,j). As indicated by the "bundle constraints" (6.3c), the total flow on any arc cannot exceed its capacity. Further, as captured by (6.3d), the model contains additional capacity restrictions on the flow of each commodity on each arc.

Observe that if the multicommodity flow problem does not contain bundle constraints, then it decomposes into r single commodity minimum cost flow problems, one for each commodity. With the presence of the bundle constraints (6.3c), the essential problem is to distribute the capacity of each arc to individual commodities in a way that minimizes overall flow costs.

We first consider some special cases. The *multicommodity maximum flow problem* is a special instance of (6.3). In this problem, every commodity k has a source node and a sink node, represented respectively by s^k and t^k. The objective is to maximize the sum of flows that can be sent from s^k to t^k for all k. Hu [1963] showed how to solve the two-commodity maximum flow problem on an undirected network in pseudopolynomial time by a labeling algorithm. Rothfarb, Shein and Frisch [1968] showed how to solve the multicommodity maximum flow problem with a common source or a common sink by a single application of any maximum flow algorithm. Ford and Fulkerson [1958] solved the general multicommodity maximum flow problem using a column generation algorithm. Dantzig and Wolfe [1960] subsequently generalized this decomposition approach to linear programming.

Researchers have proposed three basic approaches for solving the general multicommodity minimum cost flow problems: *price-directive decomposition, resource-directive decomposition* and *partitioning methods*. We refer the reader to

the excellent surveys by Assad [1978] and Kennington [1978] for descriptions of these methods. The book by Kennington and Helgason [1980] describes the details of a primal simplex decomposition algorithm for the multicommodity minimum cost flow problem. Unfortunately, algorithmic developments on the multicommodity minimum cost flow problem have not progressed at nearly the pace as the progress made on the single commodity minimum cost flow problem. Although specialized primal simplex software can solve the single commodity problem 10 to 100 times faster than the general purpose linear programming systems, the algorithms developed for the multicommodity minimum cost flow problems generally solve thse problems about 3 times faster than the general purpose software (see Ali et al. [1984]).

Network Design

We have focused on solution methods for finding optimal routings in a network; that is, on analysis rather than synthesis. The design problem is of considerable importance in practice and has generated an extensive literature of its own. Many design problems can be stated as fixed cost network flow problems: (some) arcs have an associated fixed cost which is incurred whenever the arc carries *any* flow. These network design models contain 0-1 variables y_{ij} that indicate whether or not an arc is included in the network. Typically, these models involve multicommodity flows. The design decisions y_{ij} and routing decisions x_{ij}^k are related by "forcing" constraints of the form

$$\sum_{k=1}^{r} x_{ij}^k \le u_{ij}\, y_{ij}\,, \text{ for all } (i,j)$$

which replace the bundle constraints of the form (6.3c) in the convex cost multicommodity flow problem (6.3). These constraints force the flow x_{ij}^k of each commodity k on arc (i,j) to be zero if the arc is not included in the network design; if the arc is included, the constraint on arc (i,j) restricts the total flow to be the arc's design capacity u_{jj}. Many modelling enhancements are possible; for example, some constraints may restrict the underlying network topology (for instance, in some applications, the network must be a tree; in other applications, the network might

need alternate paths to ensure reliable operations). Also, many different objective functions arise in practise. One of the most popular is

$$\text{Minimize} \sum_{k=1}^{r} \sum_{(i,j)\in A} c_{ij}^{k} x_{ij}^{k} + \sum_{(i,j)\in A} F_{ij} y_{ij}$$

which models commodity dependent per unit routing costs c_{ij}^{k} (as well as fixed costs F_{ij} for the design arcs).

Usually, network design problems require solution techniques from any integer programming and other type of solution methods from combinatorial optimization. These solution methods include dynamic programming, dual ascent procedures, optimization-based heuristics, and integer programming decomposition (Lagrangian relaxation, Benders decomposition) as well as emerging ideas from the field of polyhedral combinatorics. Magnanti and Wong [1984] and Minoux [1985, 1987] have described the broad range of applicability of network design models and summarize solution methods for these problems as well as many references from the network design literature. Nemhauser and Wolsey [1988] discuss many underlying methods from integer programming and combinatorial optimization.

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